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A BIFURCATION MODEL OF THE QUANTUM FIELD

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The paper reveals a new interpretation of the Standard Model for elementary particle physics. The approach is based on the concept of chaotic behavior applied to the gauge transformation. Following the framework of bifurcation theory, the paper provides a simple and consistent picture of lepton and boson production.

The author reveals the fractal structure of the quantum harmonic oscillator. A map of quantum field organisation is developed together with the Lagrangian formalism of the model. On this map the structure of the field appears as a "branching out" pattern.

The paper describes the internal symmetry between gravity and electromagnetism with respect to the gauge transformation. A discussion on the physical meaning of the W field and of the isospin eigenstate T_3 is included.

1. Introduction

A major achievement in the field of elementary particle physics has been the development of the Standard Model (SM). As is currently accepted, this model represents a unique synthesis of the theory of the strong force (QCD) with the theory of electroweak interaction. QCD is basically a non-Abelian gauge theory on the SU(3) color while the electroweak model successfully describes the interaction of quarks and leptons.

Although a formally consistent approach, SM is far from being completed. There are several open questions that await clarification and further extensions are yet to come. Among the issues that are not covered by the SM, the key ones are as follows:

(A) Why are there three families of gauge bosons?

(B) Why are there three flavors of neutrinos and why are they left-handed?

(C) Why six quarks and eight gluons?

(D) Is there a physical background underlying the spontaneous breaking of symmetry and is the Higgs mechanism the only way to understand the mass spectrum?

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The present paper is intended to fill in some of the remaining "holes" in the conceptual structure of the SM.

The approach is based upon the geometry of fractals and the theory of bifurcations applied to nonlinear dynamical systems. As stated, our goal is to deepen the understanding of SM without altering its fundamental construction. Consequently, our theory suggests an alternative route to SM.

Fractals have found wide application in many forefront areas of physics such as condensed matter, crystals growth, polymer statistics, fracture propagation, physics of spin systems and so on.

An analysis of the quantum harmonic oscillator performed in phase space reveals that it is possible to assign a fractal configuration to this space. Since the scale invariant properties of fractals are intimately connected with the chaotic behavior of nonlinear systems, it makes sense to further investigate how these concepts can be incorporated in the SM.

The physical foundation of this treatment lies in the following:

(A) All elementary particles (leptons, quarks and gauge bosons)–taken as solutions of the quantum field equations–are essentially nonlinear dynamical systems.

(B) The bifurcation theory–as developed by Feigenbaum in 1977–claims universality over the internal evolution of nonlinear dynamical systems.

As a result, there is a specific instability associated with the equations describing the field transformations. It is shown that this mechanism can unveil the spectrum of field quanta in a sequential manner.

The paper develops by applying the theory of bifurcations directly to the gauge transformation. The gauge transformation is the backbone of the whole SM construction. It is the nonabelian gauge invariance that makes the SM a renormalizable theory and determines the phenomenological structure of it [1].

The main benefit of this approach lies in the fact that particles appear organized in a regular and self-similar pattern. The quantum field has a structure which repeats itself from family to family. This "branching-out" layout is consistent with the nonabelian symmetries of the SM and brings into a unique picture all known bosons and fermions.

A possible extension of the bifurcation model is also outlined. This formalism explores the derivation of vacuum expectation values for the field in a manner that bypasses the Higgs mechanism. As mentioned earlier, the framework of the Higgs mechanism is not presently understood.

The paper finally examines a possible extension of the bifurcation model to include the gravitational field in the picture. The underlying physics relates to the internal symmetry of gravity and electromagnetism with respect to the gauge transformation. A discussion on the physical meaning of the W boson and of the isospin T_3 is presented. Several postulates are introduced to make

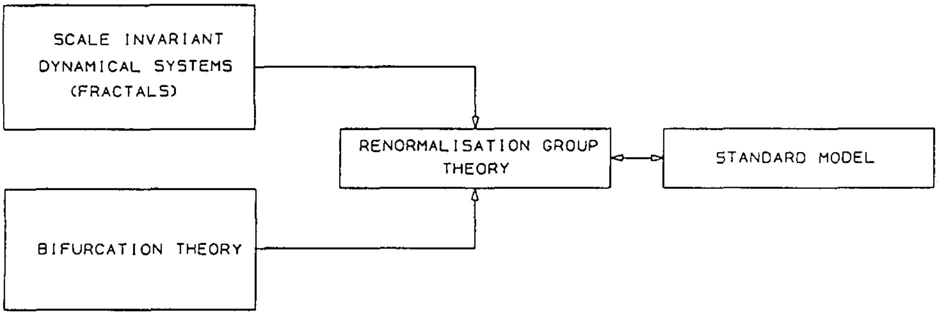

Fig. 1.

the treatment self-contained and to set its limits of validity. The rationale of each postulate is briefly reviewed below:

(P1). This postulate originates in the uncertainty principle and sets the quantum-mechanical zero point fluctuations of canonical variables. Therefore it has to be understood as a definition of the first order differential from the standpoint of the measurement process.

(P2). This postulate states the definition of a "noise-free" representation of the theory and originates also in the measurement process of canonical variables.

(P3). This postulate is a consequence of the "charge" conservation theorems from the relativistic quantum field theory.

(P4), (P5). These are transcriptions of the exclusion principle and of the Schwinger–Lüders–Pauli theorem, respectively.

Our approach is part of a general effort of theorists to broaden the knowledge basis of SM. Its relationship with the SM lies mainly in the so-called Renormalization Group ideas [1, 2] as implied by the diagram in fig. 1.

2. A fractal treatment of the harmonic oscillator

This section is an attempt to prove that the one-dimensional vibrational mode can be represented as a fractal object. As is well known, the linear harmonic oscillator is described by canonical operators p and q satisfying the commutation relation

$$[q, p] = \mathrm{i} . \tag{1}$$

The analogous classical system exhibits elliptical orbits in the phase space,

$$\frac{p^2}{2mE} + \frac{q^2}{(2E/m\omega^2)} = 1 \,, \tag{2}$$

where E is the total energy and m is the reduced mass of the particle. The area determined by the orbit boundary is always a finite number:

$$S = \int_{\mathrm{D}} p \, \mathrm{d}q = n + \tfrac{1}{2} \,, \tag{3}$$

n being a finite positive integer (Sommerfeld's quantisation rule). To evaluate the length of the orbit boundary we introduce two postulates as follows.

⟨P1⟩. The first order differential of canonical variables has the same order of magnitude as their respective fluctuations [3],

$$\mathrm{d}q = \Delta q = (2\omega)^{-1/2} \tag{4}$$

$$\mathrm{d}p = \Delta p \geqslant (2\,\Delta q)^{-1} \tag{5}$$

⟨P2⟩. The coordinate fluctuation is smaller than the coordinate itself:

$$\Delta q / q = 1/q(2\omega)^{1/2} < 1 \tag{6}$$

The above statement takes into account the fact that measurement of q is supposed to provide an acceptable level of resolution. Only this circumstance is able to generate a "noise-free" representation of the field behaviour. The length of the orbit is the line integral given by

$$L = 4 \int_0^{(2E/m\omega^2)^{1/2}} \left[1 + \left(\frac{\mathrm{d}p}{\mathrm{d}q}\right)^2\right]^{1/2} \mathrm{d}q \,. \tag{7}$$

Let us replace now the orbit slope by the ratio of coordinate and momentum fluctuations and use postulate ⟨P2⟩,

$$L \geqslant 4 \int_0^{(2E/m\omega^2)^{1/2}} [1 + \tfrac{1}{4}(\Delta q)^{-4}]^{1/2} \, \mathrm{d}q \geqslant 4 \int_0^{(2E/m\omega^2)^{1/2}} [1 + \tfrac{1}{4}(q)^{-4}]^{1/2} \, \mathrm{d}q \tag{8}$$

$$\rightarrow \frac{L}{2} \geqslant \int_0^{(2E/m\omega^2)^{1/2}} q^{-2}(1 + 4q^4)^{1/2} \, \mathrm{d}q = \int_0^1 \cdots + \int_1^{(2E/m\omega^2)^{1/2}} \cdots \,. \tag{9}$$

This definite integral can be computed starting from the general form [4]:

$$J=\int_0^1 x^{p-1}(1-x)^{n-1}(1+bx^m)^l\,\mathrm{d}x \qquad (b^2>1)\,, \tag{10}$$

where

$$p=-1\,,\quad n=1\,,\quad m=4\,,\quad l=\tfrac{1}{2}\quad\text{and}\quad [0,1]\in[0,(2E/m\omega^2)^{1/2}]\,, \tag{11}$$

which leads us to the following result:

$$L\geqslant 2\sum_{K=1}^{\infty}\frac{1}{2K}\,\frac{4^K\Gamma(4K-1)}{\Gamma(4K)}\,, \tag{12}$$

where Γ represents the factorial function

$$\Gamma(1+z)=z!\,, \tag{13}$$

such that

$$L\geqslant\sum_{K=1}^{\infty}\frac{4^K}{K}\,\frac{(4K-2)!}{(4K-1)!}=\sum_{K=1}^{\infty}\frac{4^K}{K}\,\frac{1}{4K-1}\,. \tag{14}$$

This numerical series diverges according to D'Alembert's convergence criterion [4]:

$$\lim_{K\to\infty}\frac{U_{K+1}}{U_K}=4\lim_{K\to\infty}\frac{K(4K-1)}{(K+1)(4K+3)}=4>1\,. \tag{15}$$

The result above indicates that the orbit length is a nonfinite quantity while the area (3) bounded by the same orbit remains finite. This indicates that the quantum field associated with the harmonic oscillator is suitable for a fractal description. Furthermore, since scale invariant properties of fractals are closely related to "chaos" and "strange attractors" in dynamical systems, it makes sense to attempt to fuse these concepts to SM.

3. The chaotic behavior of the global gauge transformation

Consider an isolated packet of waves representing an arbitrary free field $\phi(x)$. There are many examples of such "soliton-like" modes which can be configurated as local peaks. For instance one can take the ground state of the quantum harmonic oscillator [5]:

$$\phi^0(x) \sim \exp(-\tfrac{1}{2}x^2)\,, \tag{16}$$

where x stands for the field coordinate and the first order Hermite polynomial is

$$H_0(x) = 1\,. \tag{17}$$

Another traditional "soliton-like" wave can be introduced by performing a Fourier transformation on the space packet such that [6]

$$\phi_0(\boldsymbol{k}) \sim \int_{\mathrm{D}} \mathrm{d}^3x \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x})\phi_0(x)\,. \tag{18}$$

With a translation of the wave number,

$$\boldsymbol{k}' = \boldsymbol{k} - \boldsymbol{k}_0\,, \tag{19}$$

where

$$k_0 = 2\pi/\lambda_0\,, \tag{20}$$

the function (18) has a unique maximum at $k' = 0$ and is symmetrical with respect to the origin:

$$\begin{aligned} &\partial\phi_0/\partial k' > 0 \qquad \text{for } k' < 0\,, \\ &\partial\phi_0/\partial k' < 0 \qquad \text{for } k' > 0\,. \end{aligned} \tag{21}$$

Consider now the standard gauge transformation which leaves the structure of the theory invariant [1] and apply this to (18):

$$\phi_0(\boldsymbol{k}) \to \phi_0'(\boldsymbol{k}) = \exp(-\mathrm{i}\chi)\,\phi_0(\boldsymbol{k})\,, \tag{22}$$

where χ is equivalent to an arbitrary "phase" factor. Notice that $\exp(-\mathrm{i}\chi)$ can be thought as a rotation operator in an appropriate space such that its components are always smaller or equal than unity. Since rotations generate a group of transformations with multiplication as a composition rule, a sequence of iterations applied to (22) will not alter its form. Consequently, from a physical standpoint, there is no way of separating "a priori" a first order iteration from a nth order one given by

$$\phi_0^{(n)}(k) = \underbrace{\phi_0 \cdot (\phi_0 \cdot (\phi_0 \cdot \cdots \cdot \phi_0(k)))}_{n \text{ times}} = \exp(-\mathrm{i}n\chi)\, \phi_0(k)\,. \tag{23}$$

On the other hand, transformation (22) defined above belongs to a larger class of mappings which display chaotic behavior. Under a given set of iterations, (22) may exhibit convergence to a stable attractor or erratic divergence depending on the phase selection. It is shown that a gradual variation of the phase factor drives a sequence of bifurcations such that, for each $j \geqslant 0$, function (22) has a single unstable orbit of period 2^j [2]. On letting the phase vary beyond a critical value the cycles generation replicates itself and (22) develops unstable orbits of period $3 \cdot 2^j$. The bifurcation scenario unfolds in a manner that makes it comparable to the concept of scale invariance derived from the geometry of fractal sets.

The chaotic behavior of the gauge transformation is similar to the divergence associated with the harmonic solution of the Klein–Gordon equation

$$(\partial^\mu \partial_\mu + m^2)\phi = 0\,, \tag{24}$$

where the frequency is a complex number,

$$\omega = \omega_\mathrm{R} + \mathrm{i}\omega_\mathrm{I} = \pm\sqrt{k^2 + m^2} \qquad (k = 2\pi/\lambda)\,, \tag{25}$$

such that $\omega_\mathrm{I} = 0$ defines the marginal stability and $\omega_\mathrm{I} > 0$ generates exponentially growing amplitudes [7].

The above considerations indicate that the non-univocity of phase choice leads to an internal instability of the gauge transformation. Therefore, the degeneracy of the gauge fields may be related to a "branching-out" pattern implied by the bifurcation mechanism. As a result, the operational equivalence of $\exp(-\mathrm{i}\chi)$ and $\exp(-\mathrm{i}m\chi)$ may be regarded as the source of field architecture.

Furthermore, because gauging the elementary particles physics is the central idea of the Standard Model, we believe that a "chaotic" formalism can provide a simple and consistent description of the quantum field dynamics.

4. Postulates and conventions

To develop the frame of our approach the following statements are taken as assumptions:

⟨P3⟩. Antisymmetric fields (labeling fermions) are generated or annihilated always in pairs. Consequently, the doublets are the fundamental eigenstates of the fermionic field, while singlets represent excited eigenstates.

⟨P4⟩. Exclusion principle applies to all fermions.

⟨P5⟩. *CPT* invariance is always valid when applied to either bosons or fermions while partial symmetries (like *CP*, *P*, *T*, . . .) are not necessarily valid and may be violated.

As a result of ⟨P5⟩, *CPT* invariance operates as a restriction rule which forbids some of the field states to appear as distinct (see section 5).

As a convention, the order in which the components of a doublet are listed corresponds to the time relationship between "cause" and "effect". Therefore, if T stands for the time inversion operator and (f_1, f_2) is a generic doublet, then

$$T(f_1, f_2) = (f_2, f_1) . \tag{26}$$

Finally, because the history of field transformation may be tracked in terms of phase selection (χ) or the number of cycles accumulated within the bifurcation process (N), either one of these two variables can be taken as an independent field coordinate.

5. The internal dynamics of the quantum field

The goal of this section is to generate and explain a map of field dynamics as outlined in section 3.

If one takes the period of cycles arisen from bifurcations as an input variable (N) and the phase χ as the output, then a plot of the field architecture looks like fig. 2.

In this map V_1, V_2, V_3, . . . or V^1, V^2, . . . stand for bifurcation vertices. Let us consider a generic field of form (18) being subjected to the bifurcation mechanism. For the time being, the particular structure of the field is not relevant so $\phi_0(k)$ may be scalar, spinorial, vectorial or tensorial. In each of vertices V_j or V^j ($j \leq N$) transformation of the field must face the following "dilemma": if n_j is the number of cycles already generated and p_j is the number of new cycles, then the field behaves either symmetrically or antisymmetrically with respect to the transposition $n_j \leftrightarrow p_j$:

$$\phi(n_j, p_j) = e^{i\alpha}\phi(p_j, n_j) , \qquad e^{i\alpha} = \pm 1 . \tag{27}$$

It seems natural to assign the symmetry of the field in (27) to bosons while

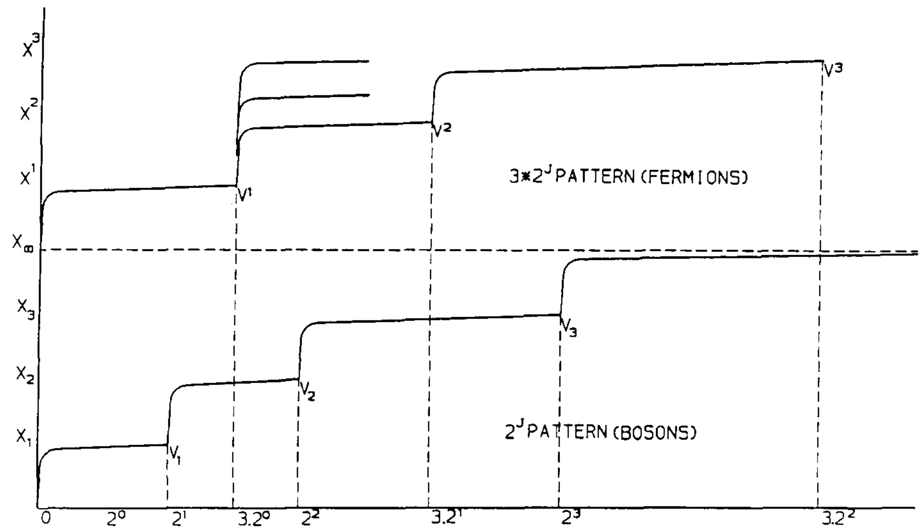

Fig. 2.

antisymmetry is associated with fermions. Since there is no privilege for either one of the two options, in each bifurcation vertex the field transformation may "jump" from the first branch of the plot to the second one. Therefore, if one considers the 2^j pattern of cycle accumulation typical for bosons and the $3 \cdot 2^j$ pattern typical for fermions, a transition boson–fermion appears to be a natural consequence of (27).

Let us discuss now each of the map vertices. Following the Standard Model, there are three symmetries to be taken into account for the gauge bosons: U(1), SU(2) and SU(3). Accordingly, three families of gauge bosons are introduced: the photon (γ), the weak triplet (W^+, W^-, W^0) and the multiplet of eight gluons ($g_1, g_2, \ldots, g_8$). The above map suggests a logical extension of this configuration as described below:

(a) In V_1 a number of 2^1 bosons are created and a natural partner for the photon (γ) could be the graviton (g).

(b) In V_2 a number of 2^2 bosons are created and the weak triplet may be replaced by two SU(2) weak triplets as follows:

$$V_2: \quad \zeta_1 = \begin{pmatrix} W^+ \\ W^- \\ W^0 \end{pmatrix}, \quad \zeta_2 = \begin{pmatrix} W^+ \\ W^- \\ B^0 \end{pmatrix}. \tag{28}$$

Here B^0 is the "extra" massive boson added to the initial family (W^+, W^-, W^0).

(c) In V_3 a number of 2^3 bosons are created and these may be identified with the gluon multiplet $(g_1, g_2, \ldots, g_8)$.

Following a similar judgement one can describe the structure of the fermionic vertices V^j.

(a) At V^1 the field triplicates because a number of $3 \cdot 2^0$ fermions are created. According to postulate $\langle P3 \rangle$ these eigenstates must be doublets. Since a transition boson–fermion is likely to take place (see (27)) and V_1 is the nearest bosonic vertex, it makes sense to assume that V^1 is filled in with ultrarelativistic doublets (neutrino type),

$$V^1: \quad (\nu_e, \bar{\nu}_e), \quad (\nu_\mu, \bar{\nu}_\mu), \quad (\nu_\tau, \bar{\nu}_\tau). \tag{29}$$

We will focus below only on the electronic neutrino branch. Assuming the *CPT* invariance (postulate $\langle P5 \rangle$), neutrino doublets appear only in a polarized state. If L and R stand for "left-hand" and "right-hand", then

$$CPT: \quad (\nu_e, \bar{\nu}_e)_L \to (\nu_e, \bar{\nu}_e)_R, \tag{30}$$

where the following line of operations was performed:

$$(\nu_e, \bar{\nu}_e)_L \xrightarrow{C} (\bar{\nu}_e, \nu_e)_L \xrightarrow{P} (\bar{\nu}_e, \nu_e)_R \xrightarrow{T} (\nu_e, \bar{\nu}_e)_R. \tag{31}$$

Consequently, the left-handed neutrinos and the right-handed ones are overlapped and only one polarized doublet has physical meaning. This conclusion agrees with the experimental data [1].

(b) At V^2 a number of $3 \cdot 2^1$ eigenstates develops. It makes sense to fill in this vertex with those leptons which make transitions from the neighbouring vertices possible. It appears that the electroweak interaction must play here a major role. A plausible configuration of V^2 may look like

$$\begin{aligned} V^2: \quad & (e^-, \bar{\nu}_e)_R, \quad (e^-, \ \ )_L, \quad (e^+, \nu_e)_L, \quad (e^+, \ \ )_R, \\ & (\ \ , e^-)_R, \quad (\ \ , e^+)_L. \end{aligned} \tag{32}$$

The first state is a SU(2) doublet while the second one is a SU(2) singlet. The reason for introducing such a singlet lies in the formal symmetry between $(e^-, \bar{\nu}_e)_R$ and $(e^-, \ \ )_L$. The component $\bar{\nu}_e$ is missing from the excited state $(e^-, \ \ )_L$ because left-handed antineutrinos are forbidden. The fourth and fifth states replicate the first and the second ones but are *not* identical with them. To check this statement one can apply a *CPT* operator such that

$$(e^-, \bar{\nu}_e)_R \xrightarrow{C} (e^+, \nu_e)_R \xrightarrow{P} (e^+, \nu_e)_L \xrightarrow{T} (\nu_e, e^+)_L, \tag{33}$$

$$(\nu_e, e^+)_L \neq (e^+, \nu_e)_L \, . \tag{34}$$

Also

$$(e^-, \quad)_L \xrightarrow{C} (e^+, \quad)_L \xrightarrow{P} (e^+, \quad)_R \xrightarrow{T} (\quad , e^+)_R \, , \tag{35}$$

$$(\quad , e^+)_R \neq (e^+, \quad)_R \, . \tag{36}$$

The fifth and sixth states enter as "mirror" images of electronic singlets as long as no privilege can be assigned to either left- or right-handed electrons. Relations (33) and (35) indicate that V^2 can be basically structured on three levels in an analogous manner as V^1, namely: V^2: (electron "up"; electron-neutrino "middle"; electron "down") because all the eigenstates listed in (32) are interchangeable with their symmetrical images. For instance,

$$(e^-, \bar{\nu}_e)_R \equiv (\nu_e, e^+)_L = T(e^+, \nu_e)_L \, , \tag{37}$$

$$(e^-, \quad)_L \equiv (\quad , e^+)_R = P(\quad , e^+)_L \, . \tag{38}$$

(c) At V^3 a number of $3 \cdot 2^2$ eigenstates develops which can be related to V^2 and V_3. Therefore, the structure of V^3 may be based on the quarks family and displays the following organisation:

$$V^3 : \begin{cases} (u_\alpha, d_\alpha)_L, & (u_\alpha, d_\alpha)_R, \quad (\bar{u}_\alpha, \bar{d}_\alpha)_L, \quad (\bar{u}_\alpha, \bar{d}_\alpha)_R \, , \\ (u_\beta, d_\beta)_L, & (u_\beta, d_\beta)_R, \quad (\bar{u}_\beta, \bar{d}_\beta)_L, \quad (\bar{u}_\beta, \bar{d}_\beta)_R \, , \\ (u_\delta, d_\delta)_L, & (u_\delta, d_\delta)_R, \quad (\bar{u}_\delta, \bar{d}_\delta)_L, \quad (\bar{u}_\delta, \bar{d}_\delta)_R \, . \end{cases} \tag{39}$$

Here (α, β, δ) is the triplet of colors carried by quarks such that

$$\alpha + \beta + \delta = 1 \, . \tag{40}$$

Let us point out that quarks enter *only* in doublets and cannot be therefore isolated as single excitations. Quark triplets appear as linear superpositions of these doublets.

Since every color has an anticolor ($\bar{\alpha}$, $\bar{\beta}$ or $\bar{\delta}$), *CPT* invariance must be operating as a prohibiting criterion for additional combinations in (39). In particular

$$(u_\alpha, d_\alpha)_L \xrightarrow{C} (u_{\bar{\alpha}}, d_{\bar{\alpha}})_L \xrightarrow{P} (u_{\bar{\alpha}}, d_{\bar{\alpha}})_R \xrightarrow{T} (u_{\bar{\alpha}}, d_{\bar{\alpha}})_R \, . \tag{41}$$

Notice here that *C* operates on the color only. If *C* would have been set to

Table I

Vertex	N	Configuration
V_1	2	$\gamma\, g$
V_2	4	$W^+\, W^0\, B^0\, W^-$
V_3	8	$g_1\, g_2, \dots, g_7\, g_8$
V^1	3	$(\nu_\mu, \bar{\nu}_\mu)\, (\nu_e, \bar{\nu}_e)\, (\nu_\tau, \bar{\nu}_\tau)$
V^2	6	$(e^-, \bar{\nu}_e)_R\, (e^-, \quad)_L\, (e^+, \nu_e)_L\, (e^+, \quad)_R\, (\_, e^-)_R\, (\quad, e^+)_L$
V^3	12	$(u_\alpha, d_\alpha)_L\, (u_\alpha, d_\alpha)_R \dots (\bar{u}_\gamma, \bar{d}_\gamma)_L\, (\bar{u}_\gamma, \bar{d}_\gamma)_R$

operate on the quark itself, then

$$(u_\alpha, d_\alpha)_L \xrightarrow{CPT} (\bar{d}_\alpha, \bar{u}_\alpha)_R \neq (\bar{u}_\alpha, \bar{d}_\alpha)_R \,. \tag{42}$$

To conclude this section let us mention that the muonic and taonic branches separated at V^1 are equivalent to the electronic branch discussed above. Therefore, the field architecture stays invariant with respect to the substitution

$$(\nu_e, e, u, d) \rightarrow (\nu_\mu, \mu, c, s) \rightarrow (\nu_\tau, \tau, b, t) \,. \tag{43}$$

All the above results can be listed in one summary table as presented in table I.

6. A Lagrangian description of the bifurcation model

In the conventional formulation of the Standard Model, the full Lagrangian contains the operators of covariant differentiation in the kinetic term and mass + interaction contributions. For the bosons the mass term is $-m^2b^2$ and the interaction term is linear in b. For the fermions both mass and interaction terms depend linearly on the mass and the interaction is also a linear expression of the four-vector current,

$$j^\mu = \bar{f}\gamma^\mu f \qquad (\mu = 0, 1, 2, 3) \,, \tag{44}$$

such that

$$\begin{aligned} L_b &= \tfrac{1}{2} D_\mu b D^\mu b - m^2 b^2 - b\rho(x^\mu) \,, \\ L_f &= \bar{f}\, i\gamma^\mu D_\mu f - \bar{f} m f + Q\bar{f}\gamma^\mu f A^\mu \,, \end{aligned} \tag{45}$$

where $L_b(L_f)$ stand for the bosonic (fermionic) lagrangian, $\rho(x^\mu)$ is the source of the scalar field while Q and A are, respectively, the charge and four-potential of the vector field.

To perform a complete description of a full Lagrangian, one would have to use a complicated formula taking into account all possible combinations between terms, such as

$$L_{\text{full}} = L_{\text{b}} + L_{\text{f}} + L_{\text{kinetic,bf}} + L_{\text{mass,bf}} + L_{\text{interaction,bf}} \,. \tag{46}$$

The full Lagrangian would have a composite structure including scalars, pseudoscalars, spinors or vectors.

For our model we will attempt to get the full Lagrangian starting from a simpler and more effective framework. To achieve this, several principles are to be considered:

(a) The approach is based upon a perturbative method.

(b) The full Lagrangian takes the phase χ as variable and the bosonic and fermionic fields as coefficients in the series expansion.

(c) The series contains powers less or equal to four. It can be shown that powers greater than four lead to infinities and are to be excluded [1].

(d) The interaction for both bosonic and fermionic fields is included in the quartic term of the series.

To derive the spectrum of the ground states arisen in a bifurcation vertex, one has to minimize the potential:

$$\partial L_{\text{int+mass}} / \partial \chi = 0 \,. \tag{47}$$

Following the standard procedure, further perturbations are defined by expanding the phase around the ground states. Accordingly, the ground states together with their perturbations fully specify the sequence of particles created as a result of bifurcation. An alternative expression for (23) is

$$\phi_0^{(n)}(\boldsymbol{k}) = \sum_{j=0}^{\infty} \frac{(-i)^j (n\chi)^j}{j!} \phi_0(\boldsymbol{k}) = \sum_{l=0}^{\infty} \phi_l(\boldsymbol{k}) \xi^l \,, \tag{48}$$

where

$$j = 2l \,, \tag{49a}$$

$$\phi_l(\boldsymbol{k}) = (-1)^l \phi_0(\boldsymbol{k}) / (2l)! \,, \tag{49b}$$

$$\xi = n^2 \chi^2 \,. \tag{49c}$$

It is natural to assume that the general form of the function potential must be a power series of the field $\phi_0(\boldsymbol{k})$:

$$V = L_{\text{int+mass}} = \sum_{n=0}^{4} a_n \phi_0^n(\boldsymbol{k}) , \qquad a_n \in \mathbb{R} , \tag{50}$$

as long as the Lagrangian has to be invariant under the gauge transformation

$$V[\phi_0(\boldsymbol{k})] \to V[\exp(-\mathrm{i}\chi)\, \phi_0(\boldsymbol{k})] . \tag{51}$$

Replace now the terms in (50) by their gauge images given by (22),

$$\phi_0^n(\boldsymbol{k}) \to \exp(-\mathrm{i}n\chi)\, \phi_0^n(\boldsymbol{k}) = \exp(-\mathrm{i}n\chi)\, \phi_0(\boldsymbol{k})\, \phi_0^{n-1}(\boldsymbol{k}) , \tag{52}$$

or

$$\phi_0^n(\boldsymbol{k}) \to \phi_0^{(n)}(\boldsymbol{k})\, \phi_0^{n-1}(\boldsymbol{k}) , \tag{53}$$

where $\phi_0^{(n)}(\boldsymbol{k})$ represents the nth iterate of $\phi_0(\boldsymbol{k})$ (23).

(50) then becomes

$$V = \sum_{n=1}^{4} a_n \phi_0^{n-1}(\boldsymbol{k})\, \phi_0^{(n)}(\boldsymbol{k}) , \tag{54}$$

or

$$V(\xi) = \sum_{n=1}^{4} a_n \phi_0^{n-1}(\boldsymbol{k}) \left(\sum_{l=0}^{\infty} \phi_l(\boldsymbol{k}) \xi^l \right) . \tag{55}$$

Because the series must stop at the quartic term the exact formula (55) gives way to an approximate one:

$$V(\xi) \sim \sum_{l=0}^{4} a\phi_l(\boldsymbol{k}) \xi^l = \sum_{l=0}^{4} b_l \xi^l . \tag{56}$$

To get an acceptable level of confidence for the use of (56) one must normalize (49c) such that

$$|\xi| < 1 . \tag{57}$$

The interaction term is the last one ($l = 4$) and b_4 represents the strength coefficient for both fermions and bosons.

The mass term is a superposition of odd and even powers. Since the description is general, we would expect a symmetrical behaviour of the mass term for bosons and an antisymmetrical one for fermions. Therefore, (56) splits into two components,

$$V_b = b_{2,\mathrm{mass}}\xi^2 + b_{4,\mathrm{int}}\xi^4 ,$$
$$V_f = b_0\xi^0 + b_{1,\mathrm{mass}}\xi + b_3\xi^3 + b_{4,\mathrm{int}}\xi^4 , \tag{58}$$

$$\left.\begin{aligned} V_{b,\mathrm{mass}}(\xi) &= V_{b,\mathrm{mass}}(-\xi) \\ V_{f,\mathrm{mass}}(\xi) &= -V_{f,\mathrm{mass}}(-\xi) \end{aligned}\right\} \quad \text{(at bifurcation vertices)} . \tag{59}$$

Under these circumstances (47) gives for bosons

$$\xi_{1,2} = \pm\sqrt{-b_2/2b_4} . \tag{60}$$

These are the vacuum expectation values of ξ and the treatment coincides with the spontaneously broken symmetry approach for the so-called Higgs field [1].

For fermions we get a cubic equation with the canonic form

$$\xi^3 + p\xi^2 + q\xi + r = 0 , \tag{61}$$

where

$$p = 3b_3/4b_4 , \qquad q = 0 , \qquad r = b_1/4b_4 \qquad (b_4 \neq 0) . \tag{62}$$

To check the nature of its solution, one has to evaluate the discriminant Q given by [8]

$$Q = \tfrac{1}{27}(-\tfrac{1}{27}p^6) + \tfrac{1}{4}(\tfrac{2}{27}p^3 + r)^2 , \tag{63}$$

or

$$Q = \tfrac{1}{27}p^3r + \tfrac{1}{4}r^2 = r(\tfrac{1}{27}p^3 + \tfrac{1}{4}r) . \tag{64}$$

As long as $b_1 \neq 0$ ($r \neq 0$) the cubic equation has three distinct roots which provide the vacuum expectation values for fermions.

The vacuum solutions correspond to the true stationary levels of the field subjected to the bifurcation mechanism. Therefore, it makes sense to think of them as of "massless" states associated with the field. Consequently, $\xi_{1,2}$ are connected with the photon and graviton (vertex V_1) while the three solutions of (61) relate to the ultrarelativistic neutrino doublets produced at vertex V^1.

Bifurcations evolve in a self-similar manner at next vertices repeating the pattern of V_1 or V^1. This is, for instance, the reason why at V_2 (and V^2) the map contains a duplicate (a triplicate, respectively) of the vacuum spectra,

$$V_2\colon \begin{cases} (W^+, W^-) = |\uparrow\rangle\,, \\ (W^0, B^0) = |\downarrow\rangle\,, \end{cases} \qquad \text{(SU(2) doublet)} \tag{65}$$

$$V^2\colon \begin{cases} [(e^-, \quad)_L\,, \quad (e^+, \quad)_R] = |\uparrow\rangle\,, \\ [(e^-, \bar{\nu}_e)_R\,, \quad (e^+, \nu_e)_L] = |0\rangle\,, \\ [(\quad, e^-)_R\,, \quad (\quad, e^+)_L] = |\downarrow\rangle\,. \end{cases} \qquad \text{(SU(2) triplet)} \tag{66}$$

As perturbations of the vacuum states, vertices V_j or V^j ($j \geq 2$) carry an inner excitation energy related to the shift from the ground level. Accordingly, each of the vertices V_j or V^j ($j \geq 2$) is the source of a unique mass spectrum. This circumstance supplies a plausible interpretation of how the mass should enter into the model.

7. Inertial frames under Lorenz invariance

The first segment of the bosonic branch (see fig. 2) represents the most familiar and simpler case of gauge invariance: the Lorenz transformation of inertial frames. It comes almost naturally to consider the covariance of the theory with respect to inertial frames as an immediate example of the gauge concept.

We will show in section 8 that an identical approach covers the gauge formalism of both electromagnetic and gravitational fields. Consequently, the Lorenz transformation is to be thought only as a first order approximation of the gauge formalism up to the first vertex (V_1).

Take two arbitrary inertial frames and let V stand for their relative linear velocity. In the Lorenz formula the rotational transposition of the coordinates operates as a 2×2 matrix,

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \cos\Psi & \sin\Psi \\ -\sin\Psi & \cos\Psi \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}, \tag{67}$$

where Ψ is the imaginary rotation angle given by

$$\begin{aligned} &V = \mathrm{i}\tan\Psi\,, \\ &\sin\Psi = -\mathrm{i}V/\sqrt{1 - V^2}\,, \qquad \cos\Psi = 1/\sqrt{1 - V^2}\,. \end{aligned} \tag{68}$$

The space–time forms a continuous scalar field whose components are physically equivalent. Following a standard representation, define the complex scalar

$$\eta' = (t' + \mathrm{i}x')/\sqrt{2} = \mathrm{e}^{-\mathrm{i}\Psi}(t + \mathrm{i}x)/\sqrt{2} = \mathrm{e}^{-\mathrm{i}\Psi}\eta \,,$$
$$\eta'^* = (t' - \mathrm{i}x')/\sqrt{2} = \mathrm{e}^{\mathrm{i}\Psi}\eta^* \,. \tag{69}$$

Note that (69) is identical with (22). Consequently, the space–time can be treated as a regular pseudoscalar field. If m is the mass associated with it, the Lagrangian depends only on $\eta^*\eta$:

$$L = \partial_\mu \eta^* \partial^\mu \eta - m^2 \eta^* \eta \qquad (\mu = 0, 1) \,. \tag{70}$$

Two conserved "charges" emerge from Lorenz invariance of the field η:

$$S_\mu = \mathrm{i}(\eta \partial_\mu \eta^* - \eta^* \partial_\mu \eta) \,, \qquad \partial_\mu S_\mu = 0 \,,$$
$$I = x^2 - t^2 \,, \qquad \partial_\mu I = 0 \qquad \text{(interval)} \,. \tag{71}$$

Let us recall now that the above relations (67)–(71) can be duplicated for the electromagnetic field under the following substitutions:

$$x^i \to A^i \qquad (i = 1, 2, 3) \,,$$
$$t \to \phi \,. \tag{72}$$

Therefore, there is no formal difference between the behavior of the space-time field (x^i, t) and the electromagnetic field (A^i, ϕ) with respect to the Lorenz transformation of inertial frames.

This circumstance gives us grounds to believe that the space–time and electromagnetic fields overlap in a gauge transformation theory. One may expect that these two fields separate at the ultrarelativistic limit, as long as $V \to c(=1)$ is the single instability associated with (68) and (69).

8. The ultrarelativistic scenario: photon–graviton split

As is well known, the Schrödinger equation describing the movement of a nonrelativistic spinless electron in an electromagnetic field,

$$[(1/2m_\mathrm{e})(-\mathrm{i}\nabla + eA)^2 - (\mathrm{i}\,\partial/\partial t + e\phi)]\,\Psi_\mathrm{e} = 0 \tag{73}$$

is left unchanged if one operates a gauge transformation on the electron wavefunction and on the field [1]:

$$\Psi'_e \to e^{-i\chi}\Psi_e \,, \tag{74a}$$

$$A'^{\mu} \to A^{\mu} - \partial^{\mu}\chi \qquad (e = 1)\,. \tag{74b}$$

We want to show that an identical picture can be developed for a particle of mass m submerged in a gravitational field. Following Einstein, a space–time wavefunction is to be attached to the particle:

$$\eta^{\mu} \leftrightarrow m \,, \tag{75}$$

which may be expressed in the standard exponential form

$$\eta^{\mu} = (\eta_0)^{\mu}\, e^{-i\rho_{\mu}} \,, \tag{76}$$

where $(\eta_0)^{\mu}$ is an amplitude and ρ_{μ} is the phase four-vector. Consequently, the equation of motion [9]

$$du_{\mu}/ds - \tfrac{1}{2}\,\partial_{\mu} g_{\nu\omega} u^{\nu} u^{\omega} = 0 \,, \tag{77}$$

where u_{μ} is the four-velocity and $g_{\nu\omega}$ the metric tensor, would not be affected if one would replace the following items:

$$\begin{aligned} &\eta^{\mu} \to \eta'^{\mu} \,, \\ &g_{\nu\omega} \to g'_{\nu\omega} \,. \end{aligned} \tag{78}$$

The presence of the particle in the gravitational field is expected to produce a small perturbation of the latter. Therefore, a weak fluctuation of the space–time geometry is to be written as

$$\eta'^{\mu} = \eta^{\mu} + \delta^{\mu}_{\eta} \,, \tag{79a}$$

$$g'_{\nu\omega} = g_{\nu\omega} + \delta g_{\nu\omega} \,. \tag{79b}$$

Taking into account (76), (79a) can be reduced at a form similar to (74a),

$$\eta'^{\mu} \to (\eta_0)^{\mu}\, e^{-i(\rho_{\mu} + \delta\rho_{\mu})} \,. \tag{80}$$

(79b) leads to a first order variation of the metric tensor given by [9]

$$g'_{\nu\omega} \to g_{\nu\omega} - (\partial_\omega \tilde{\rho}_\nu + \partial_\nu \tilde{\rho}_\omega)\,, \qquad \tilde{\rho}_\mu = \exp(-\mathrm{i}\delta\rho_\mu)\,, \tag{81}$$

which is analogous with (74b).

Because we are operating in a non-Euclidian geometry, the weak fluctuations of the space–time have to be considered as covariant perturbations. Thus, the simple derivatives in (81) are to be replaced by the covariant ones whose iterations do not alter the gauge invariance of the theory.

Since (77) originates in cancelling out the covariant derivative of the four-velocity [9, 10],

$$\mathrm{D}u^\mu = 0\,, \tag{82}$$

it is now obvious that a covariant perturbation of (82) induced by (80) would not change its form, i.e.

$$\begin{aligned} &\mathrm{D}(u^\mu + \delta u^\mu) = 0\,, \\ &\mathrm{D}(\delta u^\mu) = 0\,, \end{aligned} \tag{83}$$

or

$$\mathrm{D}(\delta u^\mu) = \mathrm{d}(\delta u^\mu) - \delta(\delta u^\mu) = 0\,, \tag{84}$$

where δu^μ are the small fluctuations of the four-velocity.

Let us investigate now a further development of our approach to the Standard Model.

As long as the electromagnetic and gravitational fields seem to have a unique background originated in the gauge transformation, it makes sense to reveal a unique formula for the covariant derivative related to these fields.

For an electrically charged particle Q submerged in an electromagnetic field A^μ the covariant derivative is [1]

$$\mathrm{D}^\mu = \partial^\mu - \mathrm{i}QA^\mu\,. \tag{85}$$

Consider an arbitrary four-vector V^μ and recall its covariant derivative for a gravitational field:

$$\mathrm{D}V^\mu = (\partial_\omega V^\mu + \Gamma^\mu_{\nu\omega} V^\nu)\,\mathrm{d}x^\omega\,, \tag{86}$$

where $\Gamma^\mu_{\nu\omega}$ are the Christoffel symbols.

For the sake of simplicity let us set

$$|\mathrm{d}x^{\omega}| = 1 \,, \tag{87}$$

and make the transposition [9]

$$V^{\nu} = V^{\mu}\delta^{\nu}_{\mu} \,, \tag{88}$$

where δ^{ν}_{μ} is the Kronecker symbol. (86) becomes

$$\mathrm{D}V^{\mu} = (\partial_{\omega} + \tfrac{1}{2}\delta^{\nu}_{\mu} g^{\mu\varepsilon}\Gamma_{\varepsilon,\nu\omega})V^{\mu} \,, \tag{89}$$

where

$$\Gamma_{\varepsilon,\nu\omega} = \partial_{\omega} g_{\varepsilon\nu} + \partial_{\nu} g_{\varepsilon\omega} - \partial_{\varepsilon} g_{\nu\omega} \,. \tag{90}$$

Here δ^{ν}_{μ} and $g^{\mu\varepsilon}$ are constants with respect to the covariant differentiation [9] and therefore they can be compared to the electrical charge in (85). Notice also that $\Gamma_{\varepsilon,\nu\omega}$ is written in the same form as the tensor of the electromagnetic field:

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \,. \tag{91}$$

The treatment outlined above suggests a physical interpretation of the weak isospin as well as a definition for the component W^{0}_{μ} of the SU(2) weak triplet $(\mathrm{W}^{\pm}, \mathrm{W}^{0})$. To achieve this, let us analyze the second term of the covariant derivative expressed in the Standard Model as [1]

$$T_2 = -\tfrac{1}{2}\mathrm{i} g_1 Y B_{\mu} \,. \tag{92}$$

The massive bosonic field B_{μ} is a linear superposition of the electromagnetic field A_{μ} and of W^{0}_{μ},

$$B_{\mu} = \sqrt{1 + [(g_1/g_2)Y_{\mathrm{L}}]^2}\, A_{\mu} + (g_1/g_2)Y_{\mathrm{L}} W^{0}_{\mu} = c_{\mathrm{a}} A_{\mu} + c_{\mathrm{w}} W^{0}_{\mu} \,, \tag{93}$$

such that (92) becomes

$$T_2 = -\mathrm{i} g_1 (Q - T_3)(c_{\mathrm{a}} A_{\mu} + c_{\mathrm{w}} W^{0}_{\mu}) \,. \tag{94}$$

In an orthogonal representation, we can conveniently cancel out some components of the scalar product,

$$Q W^{0}_{\mu} = T_3 A_{\mu} = 0 \,, \tag{95}$$

and (94) reduces to

$$T_2 = -\mathrm{i}g_1 c_\mathrm{a} Q A_\mu + \mathrm{i}g_1 c_\mathrm{w} Y T_3 W^0_\mu = T_\mathrm{E} + T_\mathrm{G} . \tag{96}$$

Since the first term is an electromagnetic contribution (according to (85)), the second term may be assigned to a gravitational one. Given the orthogonality condition, the gravitational component becomes

$$T_\mathrm{G} = -\mathrm{i}g_1 c^0_\mathrm{w} T^2_3 W^0_\mu \qquad (c^0_\mathrm{w} = 2g_1 g_2^{-1}) . \tag{97}$$

A comparison of (97) with the second term of (89) reveals the following similitudes:

$$T^2_3 \rightarrow \delta^\nu_\mu g^{\mu\varepsilon} , \quad W^0_\mu \rightarrow \Gamma_{\varepsilon,\nu\omega} . \tag{98}$$

This circumstance indicates that the isospin and the W particles are intimately connected with the gravitational field.

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Bifurcations and pattern formation in particle physics: An introductory study

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Bifurcations and pattern formation in particle physics: An introductory study

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PACS 12.60.-i – Models beyond the standard model
PACS 05.45.-a – Nonlinear dynamics and chaos
PACS 05.30.Pr – Fractional statistics systems (anyons, etc.)

Abstract – Quantum field theories lead in general to a large number of coupled nonlinear equations. Solving field equations in analytic form or through lattice-based computations is a difficult task that has been only partially successful. We argue that the theory of nonlinear dynamical systems offers valuable insights and a fresh approach to this challenge. It is suggested that universal transition to chaos in nonlinear dissipative systems provides novel answers to some of the open questions surrounding the standard model for particle physics.

Overview and motivation. – Quantum field theory (QFT) is a mature conceptual framework whose predictive power has been consistently proven in both high-energy physics and condensed-matter phenomena [1–3]. From a historical perspective, QFT represents a successful synthesis of quantum mechanics and special relativity and consists of several models. Among these, gauge theories play a leading role. The standard model (SM) is a subset of QFT whose gauge group structure includes the electroweak and strong interactions of all known elementary particles. SM is a robust theoretical framework; however, it contains some 20 adjustable parameters whose physical origin is presently unknown and whose numerical values are exclusively fixed by experiments.

Non-Abelian gauge theories are essentially nonlinear field models. Quantizing this class of models is a non-trivial effort and raises a series of theoretical challenges [4–6]. For example, no complete quantum version of classical gravity exists. Quantum chromodynamics (QCD) is considered a reliable field theory at short distances but because its coupling constant becomes large in the infrared sector, standard perturbative techniques do not apply. At present, there is no universal prescription for deriving and handling closed-form solutions of QCD field equations. This is in manifest contrast with quantum electrodynamics (QED) and the electroweak theory, where perturbative methods are applicable and analytic results possible. In general, dealing with closed-form solutions of field theories is seldom a practical alternative. For example, Heisenberg's non-perturbative quantization procedure [7,8] or Schwinger-Dyson formalism [9] lead to an infinite set of coupled differential equations which connect all orders of Green's functions. This system does not have analytic and uniquely determined solutions. In these instances, one seeks plausible assumptions that simplify the equations *or* employs suitable numerical techniques for approximation.

In its traditional form, one frequently cited shortcoming of QFT is its inherent limitation in dealing with the effect of highly unstable fluctuations *or* with a dynamics regime that is driven far away from equilibrium [10–12]. In general, pattern formation is possible in out-of-equilibrium physical systems that are *open* and *nonlinear* [13–15]. Within a *closed* system patterns may only survive as a transient and die out as a result of the relaxation towards equilibrium. It is for this reason that traditional QFT, with few notable exceptions, is largely unable to properly detect and characterize pattern formation. Recent years have shown that pattern formation is relevant to a wealth of applications ranging from reaction-diffusion processes, nonlinear optics, nanostructures and fluid mechanics to hot plasma, traffic models, epidemic spreading, transport in heterogeneous media and neural networks. [13,16,17]

Understanding non-equilibrium phenomena and pattern formation is still in its infancy. Progress in this field has benefited from tools that have been recently developed for nonlinear dynamics, bifurcation and stability theory [13,15,18–22]. Our goal here is to explore the far-from-equilibrium sector of field theory using some of these

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newly developed methods. The underlying motivation is that nonlinear dynamics brings novel insights and a practical alternative for the analysis of field equations.

The paper is organized as follows: working at a classical level, we start from a non-equilibrium "toy" model containing an Abelian gauge field coupled to a massless scalar field. The concept of universality and the emergence of the complex Ginzburg-Landau equation (CGLE) are discussed in the next section. Mass generation through period-doubling bifurcations of CGLE and the link between CGLE and the generalized exclusion statistics (GES) follow from these premises. Summary and concluding remarks are detailed in the last section.

Our contribution needs to be exclusively regarded as a preliminary research on the topic. It is neither fully rigorous nor comprehensive. We wish to convey a new qualitative view rather than an in-depth analysis of phenomena. Independent studies are required to confirm, expand or refute these tentative findings.

A "toy" model in non-equilibrium field theory. – As mentioned earlier, nonlinear field theories amount to a large set of coupled differential equations that are difficult to solve or manage through numerical approximations. The universal nature of nonlinear dynamics near the threshold of the primary instability (see, *e.g.*, [16]) suggests a shortcut route. One can start from a plausible "toy" model and generalize results to more realistic theories. One example of such a "toy" model of classical field theory describes an Abelian gauge field $a_\mu(x,t)$ in interaction with a complex massless scalar field $\varphi = \varphi_1 + i\varphi_2 = \varphi(x,t)$. The Lagrangian density reads [23]

$$\mathrm{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \left| D_\mu \varphi \right|^2 . \tag{1}$$

Here, $\mu, \nu = 0,1,2,3$ denote the space-time index, $x = (x_1, x_2, x_3)$ the spatial coordinate, $F^{\mu\nu}$ the gauge field tensor, e the coupling constant and

$$D_\mu = \partial_\mu + iea_\mu \tag{2}$$

the operator of covariant differentiation. If we take $\varphi_1 \gg \varphi_2$ for simplicity, field equations derived from (1) are given by

$$\begin{aligned} D^\mu (D_\mu \varphi) &= 0, \\ \partial^\nu F_{\mu\nu} &= 2e^2 a_\mu \varphi^2. \end{aligned} \tag{3}$$

Developing (3) yields

$$\begin{aligned} \Box\varphi &= -ie(\varphi \partial^\mu a_\mu + a_\mu \partial^\mu \varphi - a^\mu \partial_\mu \varphi) - e^2 a^\mu a_\mu \varphi, \\ \Box a_\mu &= \partial^\nu \partial_\mu a_\nu - 2e^2 \varphi^2 a_\mu, \end{aligned} \tag{4}$$

where $\Box = \partial^2/\partial t^2 - \nabla^2$ is the d'Alembert operator. To further streamline the derivation and highlight the basic argument, we proceed by assuming that the gauge field satisfies

$$\partial_\mu a_i = 0 \quad \text{for} \quad i = 1,2,3. \tag{5}$$

If a_0 denotes the temporal component of the gauge field, the system (4) can be brought to the generic form of a coupled system of partial differential equations,

$$\begin{aligned} \partial_0 \varphi &= \eta, \\ \partial_0 \eta &= f(\eta, \varphi, a_0, \xi, \nabla\eta, \ldots), \\ \partial_0 a_0 &= \xi, \\ \partial_0 \xi &= g(\eta, \varphi, a_0, \xi, \nabla\eta, \ldots), \end{aligned} \tag{6}$$

in which $f(\ldots)$ and $g(\ldots)$ are time-evolution functions and $\partial_0 = \partial/\partial t$. (6) may be presented in vector form as

$$\partial_0 \mathbf{u} = U(\mathbf{u}, \nabla\mathbf{u}, \ldots), \tag{7}$$

where $\mathbf{u} = (\eta, \varphi, \xi, a_0)$. We next posit that transition to non-equilibrium in (7) is controlled by a small external parameter $\varepsilon \ll 1$. This parameter is continuously adjustable and measures the departure from equilibrium ($\varepsilon_c = 0$). Accordingly, (7) becomes

$$\partial_0 \mathbf{u} = U(\mathbf{u}, \nabla\mathbf{u}, \ldots, \varepsilon). \tag{8}$$

The physical content of ε depends on the context of the problem at hand. In open systems ε encodes the combined effect of environmental and internal fluctuations [24]. Critical behavior in continuous dimension identifies ε with the Wilson-Fisher parameter of the regularization program ($\varepsilon = 4 - d$) [25,26]. In models involving fractional dynamics, ε characterizes the range of non-local interactions in space or the extent of temporal memory [22,27–29].

Universality and CGLE. – Non-equilibrium processes such as (8) display remarkable universality. Regardless of the specific application, macroscopic patterns that develop near the threshold of a dynamic instability are robust and largely insensitive to microscopic fluctuations [13,16,17].

Since one is familiar with the language of harmonic oscillations, we are interested in the simplest bifurcation in the dynamics of $\mathbf{u}(x,t)$ that creates oscillatory behavior. This is known as a Hopf bifurcation and represents the simplest transition that leads from a focus point to a periodic behavior. As the bifurcation point is approached, the focus point becomes unstable and gives rise to a harmonic limit cycle. CGLE is a *universal* model that holds for all pattern-forming systems undergoing a Hopf bifurcation [13,16]. The theory of the reduction to CGLE from generic systems of autonomous nonlinear equations such as (8) has been developed by several authors. The derivation of CGLE for a $(1+1)$-dimensional system starts from the ansatz

$$\mathbf{u}(x,t) = \mathbf{u}_0 + A(\tilde{x}, \tilde{t}) \exp[i(k_c x - \Omega_c t]\mathbf{u}_1 + \text{c.c.}, \tag{9}$$

where $\tilde{x}$, $\tilde{t}$ represent slow variables and k_c, Ω_c are critical values in wave number and frequency spaces. Replacing in (8), dropping the tildes and expanding in power series of the small parameter $\tilde{\varepsilon}=\varepsilon-\varepsilon_c$ leads to CGLE in its standard form

$$\partial_t A = A + (1+ic_1)\nabla^2 A - (1-ic_3)\left|A\right|^2 A. \tag{10}$$

Here,

$$A(x,t)=\rho(x,t)\exp[-i\Phi(x,t)] \tag{11}$$

is a complex-valued amplitude defining the slow modulation in space and time of the underlying periodic pattern. The real parameters c_1, c_3 denote the linear and nonlinear dispersion parameters, respectively. The limit $c_1, c_3 \to 0$ corresponds to the real Ginzburg-Landau equation, whereas $c_1^{-1},\ c_3^{-1} \to 0$ recovers the nonlinear Schrödinger equation.

Higgs-free generations of particle masses. – Among the simplest coherent structures generated by CGLE are plane-wave solutions having the form [13,16]

$$\begin{gathered} A(x,t)=A_0\exp[-i(qx+mt)]+\text{c.c.},\\ A_0=\sqrt{1-q^2}. \end{gathered} \tag{12}$$

The frequency m satisfies the dispersion equation

$$m_q = c_1 q^2 - c_3(1-q^2) \tag{13}$$

and $q\in[-1,1]$ represents the phase gradient of the complex amplitude (12),

$$q=-\nabla\left|\Phi\right| \tag{14}$$

Linear stability analysis of (12) reveals that plane waves having a wave number larger than the so-called Eckhaus threshold

$$q_E=\sqrt{\frac{1-c_1c_3}{2(1+c_3^2)+1-c_1c_3}} \tag{15}$$

are unstable with respect to the long-wavelength modulation. In particular, a vanishing Eckhaus threshold leads to the Benjamin-Feir-Newell (BFN) instability criterion (A.1)

$$c_1c_3=1. \tag{16}$$

The dispersion equation (13) has two complementary limits: $q=\pm1$ ($A_0=0$) and $q=0$ ($A_0=\pm1$). Arguments presented in appendix A suggest a natural identification of these two modes with fermion and electroweak gauge boson fields, respectively. Equation (14) implies that fermions have a non-vanishing and uniform phase gradient $\nabla\Phi\neq0$, whereas gauge bosons have a uniform phase and a vanishing phase gradient $\nabla\Phi=0$. Although we have started from a massless model, from (13) and (16) it follows that both these modes acquire non-vanishing masses. In non-dimensional form and near the BFN instability, the two sets of masses are

$$\begin{aligned} m_\pm &= c_1,\\ m_0 &= -c_3, \end{aligned} \tag{17a}$$

such that

$$m_\pm = \left|m_0\right|^{-1}. \tag{17b}$$

Table 1: Actual *vs.* predicted mass scaling ratios for $\bar{\delta}=3.9$.

Parameter ratio	Behavior	Actual	Predicted
m_u/m_c	$\bar{\delta}^{-4}$	3.365×10^{-3}	4.323×10^{-3}
m_c/m_t	$\bar{\delta}^{-4}$	3.689×10^{-3}	4.323×10^{-3}
m_d/m_s	$\bar{\delta}^{-2}$	0.052	0.066
m_s/m_b	$\bar{\delta}^{-2}$	0.028	0.066
m_e/m_μ	$\bar{\delta}^{-4}$	4.745×10^{-3}	4.323×10^{-3}
m_μ/m_τ	$\bar{\delta}^{-2}$	0.061	0.066
M_W/M_Z	$\left(1-\bar{\delta}^{-1}\right)^{1/2}$	0.8823	0.8623

It is known that plane-wave solutions consist of both positive and negative frequencies. Because mass is positive definite, in what follows we are limiting the discussion to the cases $c_1>0$ and $c_3<0$.

The Feigenbaum-Sharkovskii-Magnitskii (FSM) paradigm. – The FSM paradigm of universal transition to chaos in nonlinear dissipative systems is briefly detailed in appendix B. Extensive numerical data [20,21] show that both parameters of linear and nonlinear dispersion c_1, c_3 of (17a) are distributed in a geometric progression, that is

$$\begin{aligned} c_{1,n} &= c_{1,\infty}+K_1\bar{\delta}^{-n},\\ c_{3,n} &= c_{3,\infty}+K_2\bar{\sigma}^{-n}, \end{aligned} \tag{18}$$

where $\bar{\delta}$, $\bar{\sigma}$ are scaling constants and $n=1,2,3\ldots$ represents the number of tori accumulated through bifurcations. Since $K_1, K_2, c_{1,\infty}$ and $c_{3,\infty}$ are independent of n, they can be both absorbed into a redefinition of masses. We have, accordingly,

$$\begin{aligned} m_n^* &= \frac{1}{K_1}(m_{\pm,n}-c_{1,\infty}),\\ M_n &= \frac{1}{K_2}(m_{0,n}-c_{3,\infty}). \end{aligned} \tag{19}$$

The ratios of two arbitrary masses in the bifurcation sequence take the form

$$\begin{aligned} \frac{m_n^*}{m_{n+p}^*} &= \bar{\delta}^p,\\ \frac{M_n}{M_{n+p}} &= \bar{\sigma}^p, \end{aligned} \tag{20}$$

in which $p=2^k,\ k=1,2,3\ldots$. Based on (17) it can be concluded that, near the BFN instability, the two scaling constants are linked to each other.

Analysis of the Renormalization Group flow for the real Ginzburg-Landau equation leads to the following relationship between $\bar{\delta}$ and $\bar{\sigma}$ [28]:

$$1-\left(\frac{M_1}{M_2}\right)^2 = 1-\left(\bar{\sigma}^1\right)^2 \approx \frac{1}{\bar{\delta}}, \tag{21}$$

where $M_1=M_W,\ M_2=M_Z$ are vector boson masses. Table 1 shows a side-by-side comparison between predictions inferred from (20) and experiment, where $\bar{\delta}=3.9$

Table 2: Actual values of elementary particle masses.

Parameter	Value	Units
m_u	2.12	MeV
m_d	4.22	MeV
m_s	80.90	MeV
m_c	630	MeV
m_b	2847	MeV
m_t	170,800	MeV
M_W	80.46	GeV
M_Z	91.19	GeV

represents the numerical value of the scaling constant that best fits laboratory data [30]. Actual values of particle masses, computed at the reference scale given by the mass of the top quark [31], are listed in table 2. Note that the choice of the mass scale is completely arbitrary since (20) involves ratios of consecutive masses.

CGLE and generalized exclusion statistics. – Dispersion relation (13) indicates that plane-wave solutions of CGLE interpolate between gauge boson states ($q=0$) and fermion states ($q=\pm 1$). From (13) and (14) it follows that the *spin* associated with an arbitrary mixed state is given by[1]

$$\sigma = 1 - \frac{(\nabla\Phi)^2}{2}. \qquad (22)$$

From this standpoint, CGLE is remarkably similar to the framework describing quantum fractional statistics in condensed-matter physics. In what follows we briefly discuss this analogy. The generalized exclusion statistics (GES) is motivated by the properties of quasi-particles occurring in the fractional quantum Hall effect [32,33]. Consider a thermodynamic ensemble of N identical particles. Let d represent the dimension of the one-particle Hilbert space obtained by fixing the coordinates of the remaining $N-1$ particles. The statistics of a particle is defined by the so-called Haldane's parameter g,

$$g = -\frac{\partial d(N)}{\partial N} \approx -\frac{d(N+\Delta N) - d(N)}{\Delta N}. \qquad (23)$$

Because any given state can be populated by any number of bosons, $d(N+\Delta N) = d(N)$ and hence $g=0$. By contrast, the Pauli exclusion principle restricts fermions to $g=1$. Quasi-particles with mixed statistics are characterized by an intermediate value of g and are said to satisfy a generalized exclusion principle. In this case, it can be shown that thermodynamic quantities such energy, heat capacity or entropy can be expressed in factorized form. In particular, the energy of the quasi-particle ensemble is given by

$$E(g) = gE(1) + (1-g)E(0). \qquad (24)$$

[1]Strictly speaking, spin is a concept that is valid only in a quantum or semi-quantum context. Since our analysis is carried out at the classical level, (22) is meant to simply denote a numerical attribute of plane waves dependent on the wave number q.

Table 3: Comparison between CGLE and GES.

CGLE	GES
$q = -\dfrac{\partial\lvert\Phi\rvert}{\partial x}$	$g = -\dfrac{\partial d}{\partial N}$
$m_q = c_1 q^2 - c_3(1-q^2)$	$E_g = gE(1) + (1-g)E(0)$

An example of this type of objects is offered by *anyons*, quasi-particles that exist in two dimensions and carry fractional charges. When two particles of a system of bosons are exchanged, the phase of the system remains unchanged, whereas for a system of fermions it changes by exactly π. Exchanging two anyons results in a phase factor that falls between zero and π. Anyons play a key role in the fractional quantum Hall effect and high-temperature superconductivity [32,33].

A short comparison between plane-wave solutions of CGLE and GES is included in table 3.

Summary and conclusions. – This brief report has been motivated by recent advances in nonlinear dynamics and complexity theory. Exploiting the universal theory of transition to chaos in nonlinear dissipative systems, we have found that:

a) particles acquire mass as plane-wave solutions of CGLE, without reference to the hypothetical Higgs scalar or to a particular symmetry breaking mechanism. As of today, the reality of the Higgs doublet and nature of electroweak symmetry breaking are issues that remain unsettled.

b) Starting from a basic model of Abelian gauge bosons in interaction with scalar fields, CGLE leads to a natural separation of heavy *non-relativistic modes* ($q=0$) from light *relativistic modes* of maximal group velocity ($q=\pm 1$). The most straightforward interpretation of this result is that the first group of modes corresponds to electroweak gauge bosons and the second group to fermions.

c) A direct connection may be set up between GES in condensed-matter physics and the dispersion relation (13) corresponding to $q \neq 1$. Although different in methodology and content, both GES and CGLE point out that fractional quantum statistics and non-equilibrium field theory enable a *dynamic unification* of gauge bosons and fermions as particles with arbitrary spin. This is in contrast with super-symmetry and related models (see, *e.g.*, [34]) which are based on extended symmetry groups and pay virtually no attention to nonlinear dynamics of underlying fields.

We close this section with two short remarks: 1) the approach developed here is based on *classical* field theory. Needless to say, a realistic model cannot ignore the quantum nature of fields evolving in four-dimensional space-time. However, as previously pointed out, future results are not expected to substantially deviate from these initial findings because of *universality arguments* related to nonlinear dynamics of (8) and CGLE (see, *e.g.*, [13,18,20]); 2) although our approach bypasses the conventional Higgs mechanism, it still remains compatible with it. The standard model asserts that particle masses are generated through electroweak symmetry breaking and are attributed to the Yukawa couplings of the fermions (g_f) to the Higgs condensate (v_{H^0}). The ratio of two arbitrary fermion masses in the spectrum is given by

$$\frac{m_f}{m_{f'}} = \frac{g_f(\mathrm{v}_{H^0})}{g_{f'}(\mathrm{v}_{H^0})} = \frac{g_f}{g_{f'}}. \tag{25}$$

It follows that the mass hierarchy shown in table 1 may be simply interpreted as reflecting the hierarchy of the corresponding Yukawa couplings.

Future research may be focused on a deeper understanding of pattern formation and its ramifications in the realm of SM and beyond. Of key interest is the emergence of novel states in the TeV range of particle physics. This probing energy will become accessible in the near future at the large hadron collider and other accelerator sites [35].

Appendix A. – The two dispersion parameters of CGLE are subject to the following dynamic constraints [13,16,17]:

a) the Benjamin-Feir-Newell (BFN) criterion states that stability becomes borderline for

$$c_1 c_3 = 1; \tag{A.1}$$

b) using (13), the group velocity of the plane-wave solutions is given by

$$\mathrm{v}_g = 2q(c_1 + c_3). \tag{A.2}$$

Compliance with relativity bounds (A.2) to a constant that represents the normalized value of light speed *in vacuo*. It is clear that $q = 0$ represents a *slow mode* (massive gauge boson), while $q = \pm 1$ describes the *fastest mode* (relativistic fermions). Masses associated with these modes are supplied by (17). From the BFN criterion it follows that the borderline value of the normalization constant $Q = \frac{\mathrm{V}_{g,\max}}{2}$ can be determined from

$$\begin{gathered} c_1 = \frac{Q \pm \sqrt{Q^2 - 4}}{2} \Rightarrow Q \geqslant 2, \\ c_3 = \frac{1}{c_1}. \end{gathered} \tag{A.3}$$

Equations (A.1) and (A.2) imply that, close to the border of BFN instability, gauge boson and fermion masses scale as dual entities. This finding is consistent with the behavior of the last entry in table 1.

Appendix B. – Consider the following boundary value problem for CGLE in $1+1$ space-time dimensions [20,21,36]:

$$\begin{aligned} &\partial_t A = A + (1 + ic_1)\partial_x^2 A - (1 - ic_3)\,|A|^2\,A, \\ &\partial_x A(0,t) = \partial_x A(L,t) = 0, \quad A(x,0) = A_0(x), \\ &0 \leqslant x \leqslant L,\ 0 \leqslant t \leqslant \infty. \end{aligned} \tag{B.1}$$

This model can be reduced to a three-dimensional system of nonlinear ordinary differential equations with the help of the Galerkin few-modes approximation:

$$A(x,t) \approx \sqrt{\xi(t)}\exp[i\theta_1(t)] + \sqrt{\eta(t)}\exp[i\theta_2(t)]\cos\left(\frac{\pi}{L}x\right) \tag{B.2}$$

in which

$$\begin{aligned} \partial_t \xi &= f_1(\xi, \eta, \theta, c_1, c_3, L), \\ \partial_t \eta &= f_2(\xi, \eta, \theta, c_1, c_3, L), \\ \partial_t \theta &= f_3(\xi, \eta, \theta, c_1, c_2, L) \end{aligned} \tag{B.3}$$

with $\theta(t) = \theta_2(t) - \theta_1(t)$. It can be shown that the transition to chaos in (B.3) occurs through a *sequential cascade of bifurcations* in three separate stages. This cascade starts with the Feigenbaum scenario of period-doubling bifurcations of stable cycles, followed by the Sharkovskii subharmonic cascade and ending with the Magnitskii cascade of stable homoclinic cycles.

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Complexity in quantum field theory and physics beyond the standard model

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Abstract

Complex quantum field theory (abbreviated c-QFT) is introduced in this paper as an alternative framework for the description of physics beyond the energy range of the standard model. The mathematics of c-QFT is based on fractal differential operators that generalize the momentum operators of conventional quantum field theory (QFT). The underlying premise of our approach is that c-QFT contains the right analytical tools for dealing with the asymptotic regime of QFT. Canonical quantization of c-QFT leads to the following findings: (i) the Fock space of c-QFT includes *fractional numbers* of particles and antiparticles per state, (ii) c-QFT represents a generalization of topological field theory and (iii) classical limit of c-QFT is equivalent to field theory in *curved space–time*. The first finding provides a field-theoretic motivation for the transfinite discretization approach of El-Naschie's $\varepsilon^{(\infty)}$ theory. The second and third findings suggest the dynamic unification of boson and fermion fields as particles with fractional spin, as well as the close connection between spin and space-time topology beyond the conventional physics of the standard model.

1. Introduction and motivation

Quantum field theory (QFT) is an approximate description of particle phenomena occurring in an energy range below few hundred GeV. For this reason, QFT is considered an *effective field theory* which deliberately ignores the substructure and the degrees of freedom observable above this upper bound [1]. A number of recent studies have suggested, from a variety of standpoints, that physics in the TeV regime of QFT may be a manifestation of *complex dynamics* [2–9,11,16–19]. For example, it has been argued that the onset of large and persistent vacuum fluctuations, along with strong-gravity effects emerging from the short-distance behavior of QFT, warrant the passage from the standard tools of classical calculus to fractional calculus [16]. In general, use of conventional differential operators rests on the tacit assumption that a clear separation exists between the macroscopic and the microscopic levels of physical description. Implicit in this assumption is the condition that dynamical processes on the microscopic scale are *stable*. If this condition fails to be true, dynamical instabilities can develop on arbitrarily long time-scales and the macroscopic description of phenomena in terms of ordinary differential operators breaks down [13,15]. Such a scenario may be typical for

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physics in the TeV regime where far-from equilibrium statistical processes are expected to dominate. Let us briefly elaborate on this point with the help of an idealized quantum-mechanical experiment. Consider an isolated two-state quantum system whose state vector $|\psi\rangle$ at time $t = 0$ is given by

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle \tag{1}$$

where $|0\rangle$ and $|1\rangle$ denote two orthogonal states and c_0, c_1 are complex numbers. Assume that the quantum vacuum, acting as source of large and steady fluctuations, may be modeled as a two-state reservoir with vectors $|\mathrm{v}_0\rangle$ and $|\mathrm{v}_1\rangle$. Bring the quantum vacuum in contact with the system at some instant $t_0 > 0$ and maintain the contact for an interval $t_{\mathrm{INT}} > t_0$. The coupling of the two objects through unitary evolution leads to a time-dependent state

$$|\psi(t)\rangle = c_0|0\rangle \otimes |\mathrm{v}_0\rangle + c_1|1\rangle \otimes |\mathrm{v}_1\rangle \tag{2}$$

where $\otimes$ stands for the tensor product and $t_{\mathrm{INT}} > t > t_0$. It is seen that the system and vacuum become *entangled* on a time-scale commensurate with t_{INT} and one can no longer treat (1) as describing an object with a well-defined quantum identity. In contrast to the low-energy regime of quantum theory, the high-energy dynamics of the vacuum is characterized in general by a large number of time-scales that are not reducible to a single average through coarse-graining. It follows that the ensemble system + vacuum evolves on *multiple scales.* This line of reasoning reproduces, in essence, the statistical physics argument for replacing ordinary derivatives and integrals with fractal operators [13].

To the best of our knowledge, this work represents the first attempt to build a field theory on the basis of fractional differential and integral operators (c-QFT). Its main goal is to develop canonical quantization of c-QFT and evaluate its ramifications in connection with future extensions of QFT. We caution that our contribution is meant to serve as an informal introduction and not as a rigorous and comprehensive treatment of the topic.

The paper is organized in the following way: Sections 3 and 4 introduce canonical quantization for c-QFT of free scalar and Dirac fields. The link between Chern–Simons field theory and c-QFT is examined in Section 5. Section 6 is devoted to a brief discussion on the dual aspect of c-QFT and general relativity. Last section presents a short summary of results and future prospects.

2. Notation, assumptions and conventions

We introduce here the main notations and assumptions that underlie the remainder of the paper

(i) The summation convention is applied on repeated upper and lower indices and Planck's constant is set to $\hbar = 1$.

(ii) Motivated by the growing evidence for complexity in field theory, our focus is the behavior of fractional *dynamical systems* [10]. These systems are characterized by non-integer powers of generalized coordinates and momenta

$$q^\alpha \doteq |q|^\alpha, \quad p^\alpha \doteq |p|^\alpha \tag{3}$$

in which $\alpha > 0$. As stated, we study the dynamics of free fractional scalar and Dirac fields. To avoid cluttering the notation, the corresponding field variables are respectively designated as

$$\varphi \doteq q^\alpha, \quad \psi \doteq q^\alpha \tag{4}$$

(iii) The hat symbol "^" is used to indicate operators.

(iv) Analysis is limited to real or complex functions of the dimensionless field variable $q \geqslant 0$ for which fractional derivatives and integrals exist. We adopt hereafter the regularized expression for fractional derivative [20,21]

$$D^\alpha[f(q)] \doteq \frac{\partial^\alpha[f(q)]}{\partial q^\alpha} \doteq \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{q} \frac{\partial f(\xi)}{\partial \xi} \frac{\mathrm{d}\xi}{(q-\xi)^\alpha} \tag{5}$$

where $0 < \alpha < 1$. The fractional momentum operator is introduced in Appendix A by analogy with conventional formulation of quantum mechanics. It may be shown that fractional momentum is linear and hermitean. The latter property follows from an extended definition of the conjugate operator, as detailed in (A7)–(A10).

(v) The generalized Lagrangian for a classical fractional system depending on n fields, their fractional derivatives of orders ω_l and n locally defined exponents $\alpha_l(x)$ $(l = 1, 2, \ldots, n)$ is defined by

$$L_G^{\alpha_1(x),\alpha_2(x)\ldots,\alpha_n(x)} \doteq L_G(q_1^{\alpha_1(x)}, q_2^{\alpha_2(x)}, \ldots, q_n^{\alpha_n(x)}; D^{\omega_1} q_1^{\alpha_1(x)}, D^{\omega_2} q_2^{\alpha_2(x)}, \ldots, D^{\omega_n} q_n^{\alpha_n(x)}; t) \tag{6}$$

(vi) The commutator and anti-commutator for any pair of arbitrary operators $(\hat{f}, \hat{g})$ are, respectively

$$[\hat{f}, \hat{g}] \doteq \hat{f}\hat{g} - \hat{g}\hat{f} \tag{7}$$

$$\{\hat{f}, \hat{g}\} \doteq \hat{f}\hat{g} + \hat{g}\hat{f} \tag{8}$$

(vii) State vectors and inner products are formulated using Dirac notation.
(viii) The vacuum state is considered empty and is labeled with the zero-particle ket$|0\rangle$.
(ix) Dynamical processes described by c-QFT are Markovian and, as such, have no time memory.
(x) Greek letters μ, ν, $\sigma = 0, 1, 2, 3$ denote space–time indices whereas roman letters $i, j, k = 1, 2, 3$ label the set of three spatial coordinates.

3. Scalar bosons in c-QFT

The classical Lagrangian for the free scalar field theory in 3 + 1 dimensions reads [14,22]

$$L = \partial^{\mu}\varphi\partial_{\mu}\varphi - m^2\varphi^2 \tag{9}$$

and leads to the following expression for the field momentum:

$$\pi = \frac{\partial L}{\partial\left(\frac{\partial\varphi}{\partial t}\right)} = \frac{\partial\varphi}{\partial t} \tag{10}$$

It is known that the standard technique of canonical quantization promotes a classical field theory to a quantum field theory by converting the field and momentum variables into operators. To gain full physical insight with minimal complications in formalism, we work below in 0 + 1 dimensions. Define the field and momentum operators as

$$\begin{aligned} &\varphi \to \hat{\varphi} \doteq \varphi \\ &\pi \to \hat{\pi}^{\alpha} \doteq -\mathrm{i}\frac{\partial^{\alpha}}{\partial|\varphi|^{\alpha}} \equiv -\mathrm{i}D^{\alpha} \end{aligned} \tag{11}$$

Without loss of generality, we set $m = 1$ in (9). The Hamiltonian becomes

$$H \to \widehat{H}^{\alpha} = -\frac{1}{2}D^{2\alpha} + \frac{1}{2}\varphi^2 = \frac{1}{2}\left(\hat{\pi}^{2\alpha} + \varphi^2\right) \tag{12}$$

The state of the field in the Schrödinger representation is described by a complex-valued wavefunction $\Psi(\varphi) = \langle\varphi|\Psi\rangle$ whose conjugate-square is the probability density for φ. This wavefunction evolves according to the time-dependent Schrödinger equation

$$\mathrm{i}\partial_t\Psi(\varphi) = \widehat{H}^{\alpha}\Psi(\varphi) \tag{13}$$

The commutation relations corresponding to (11) may be written as (per Appendix B)

$$\begin{aligned} &[\varphi, \varphi] = 0 \\ &[\hat{\pi}^{\alpha}, \hat{\pi}^{\alpha}] \doteq [D^{\alpha}, D^{\alpha}] = 0 \\ &[\hat{\varphi}, \hat{\pi}^{\alpha}] = \mathrm{i}\alpha\hat{\pi}^{(\alpha-1)} \end{aligned} \tag{14}$$

By analogy with the standard treatment of harmonic oscillator in quantum mechanics, it is convenient to work with the destruction and creation operators defined through [14,23]

$$\begin{aligned} &\hat{a}^{\alpha} \doteq \frac{1}{\sqrt{2}}[\hat{\varphi} + \mathrm{i}\hat{\pi}^{\alpha}] \\ &\hat{a}^{+\alpha} \doteq \frac{1}{\sqrt{2}}[\hat{\varphi} - \mathrm{i}\hat{\pi}^{\alpha}] \end{aligned} \tag{15}$$

Straightforward algebra shows that these operators satisfy the following commutation rules:

$$\begin{aligned} &[\hat{a}, \hat{a}] = [\hat{a}^{+\alpha}, \hat{a}^{+\alpha}] = 0 \\ &[\hat{a}^{+\alpha}, \hat{a}^{\alpha}] = \mathrm{i}[\hat{\varphi}, \hat{\pi}^{\alpha}] = -\alpha\hat{\pi}^{(\alpha-1)} \end{aligned} \tag{16}$$

The second relation in (16) leads to

$$\widehat{H}^{\alpha} = \hat{a}^{+\alpha}\hat{a}^{\alpha} + \frac{1}{2}\alpha\hat{\pi}^{(\alpha-1)} \tag{17}$$

In the limit $\alpha \to 1$ we recover the quantum mechanics of the harmonic oscillator, namely

$$\widehat{H} = \hat{a}^{+}\hat{a} + \frac{1}{2} \tag{18}$$

Next, consider the commutator $[\widehat{N}^{\alpha}, \hat{a}^{+}]$, where $\widehat{N} \doteq \hat{a}^{+}\hat{a}$ designates the number operator. We obtain

$$[\widehat{N}^{\alpha}, \hat{a}^{+\alpha}] = \alpha\hat{\pi}^{(\alpha-1)}\hat{a}^{+\alpha} \tag{19}$$

Thus

$$\widehat{N}^{\alpha}\hat{a}^{+\alpha}|0\rangle = (\hat{a}^{+\alpha}\widehat{N}^{\alpha} + [\widehat{N}^{\alpha}, \hat{a}^{+\alpha}])|0\rangle = \alpha\hat{\pi}^{(\alpha-1)}\hat{a}^{+\alpha}|0\rangle \tag{20}$$

The eigenvalue equation corresponding to the above relation has the form

$$D^{(\alpha-1)}\hat{a}^{+\alpha}|0\rangle = \lambda^{(\alpha-1)}\hat{a}^{+\alpha}|0\rangle = \frac{\mathrm{i}\lambda}{\alpha}\hat{a}^{+\alpha}|0\rangle \tag{21}$$

and it is solved in Appendix C. Under the most general circumstances, λ_n (*n=1,2,...*) form a set of real numbers that may be cast in a fractional form. As a result, we are led to conclude that $\hat{a}^{+\alpha}|0\rangle$ represents an eigenvector of the number operator having fractional eigenvalues. Stated differently, the action of the creation operator $\hat{a}^{+\alpha}$ on the empty vacuum is to produce a particle that carries a fractional quantum of energy. Following the general arguments of Section 1), we may call these fractional excitations of the scalar field "*complexons*". According to Appendix C, since λ_n form a discrete set of eigenvalues, it is appropriate to regard the complexon as a *fractional particle with a discrete energy spectrum*. We close this section with the observation that, on account of (17) and (18), the term $\frac{1}{2}\alpha\hat{\pi}^{(\alpha-1)}$ plays the role of a *zero-point operator*. In contrast with conventional quantum theory, it is apparent that the dynamical contribution of the background vacuum in c-QFT amounts to more than a constant additive term to the Hamiltonian.[1]

4. Fermions in c-QFT

The classical Dirac equation describing free fermion fields in $3+1$ dimension is [14,22]

$$(\mathrm{i}\gamma^{\mu}\partial_{\mu} - m)\psi = 0 \tag{22}$$

where γ^{μ} are 4×4 matrices given by

$$\gamma^{0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^{i} = \begin{pmatrix} 0 & \sigma^{i} \\ -\sigma^{i} & 0 \end{pmatrix} \tag{23}$$

and ψ is a 4-component spinor which transforms under the spin 1/2 representation of the Lorentz group. The Dirac equation may be derived from the Lagrangian

$$L_{\mathrm{D}} = \frac{\mathrm{i}}{2}\left[\bar{\psi}\gamma^{\mu}(\partial_{\mu}\psi) - (\partial_{\mu}\bar{\psi})\gamma^{\mu}\psi\right] - m\bar{\psi}\psi \tag{24}$$

in which the adjoint spinor is defined as

$$\bar{\psi} \doteq \psi^{+}\gamma^{0} \tag{25}$$

To simplify the formalism and capture the essentials of the argument, we choose to work again in $0+1$ dimensions and set $m=1$. Let the spinor field be expanded in a basis containing the eigenstates of γ^{0} that is

$$\psi = \psi_{+}|+\rangle + \psi_{-}|-\rangle \tag{26}$$

where

$$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$\langle +| = (1 \quad 0) \quad \langle -| = (0 \quad 1) \tag{27}$$

The conjugate momentum of the spinor field is

$$\Pi = \frac{\partial L_{\mathrm{D}}}{\partial\left(\frac{\partial\psi}{\partial t}\right)} = \mathrm{i}\psi^{+} \tag{28}$$

[1] It is tempting to speculate that the emergence of zero-point operator as a dynamical generator of vacuum structure might account for the existence of dark matter in the universe. As it is known, the source and physical attributes of dark matter are currently unanswered questions in particle physics and cosmology.

Consider now the coordinate Schrödinger representation for Dirac fields, whereby an arbitrary state $|\Phi\rangle$ is represented by the wavefunction $\Phi(\psi) = \langle\psi|\Phi\rangle$. By analogy with the previous section, we cast Dirac field theory in the operator language. Let us take the state $|\psi\rangle$ to be an eigenstate of the field operator $\hat{\psi}$ with eigenvalue ψ

$$\hat{\psi}|\psi\rangle = \psi|\psi\rangle \tag{29}$$

The field conjugate momentum is then

$$\hat{\Pi} = \mathrm{i}\hat{\psi}^{+} = \mathrm{i}\frac{\partial}{\partial\psi} \tag{30}$$

Creation and destruction operators are introduced as follows [14]:

$$\begin{aligned} \hat{\psi} &= \hat{b}|+\rangle + \hat{c}^{+}|-\rangle \\ \hat{\psi}^{+} &= \hat{b}^{+}\langle+| + \hat{c}\langle-| \\ \hat{\bar{\psi}} &= \hat{b}^{+}\langle+| - \hat{c}\langle-| \end{aligned} \tag{31}$$

Here, $\hat{b}$ and $\hat{c}$ are the fermion and antifermion destruction operators, whereas $\hat{b}^{+}$ and $\hat{c}^{+}$ denote the fermion and antifermion creation operators. It is known that, to ensure that the total fermion energy is positive-definite, Dirac field theory is formulated using anticommutators rather than commutators [14,22]. The momentum anti-commutator is given by

$$\{\hat{\psi}, \hat{\Pi}\} \equiv \{\hat{\psi}, \mathrm{i}\hat{\psi}^{+}\} = \mathrm{i}[\{\hat{b}, \hat{b}^{+}\} + \{\hat{c}, \hat{c}^{+}\} - 1] \tag{32}$$

Here, according to (8)

$$\{\hat{\psi}, \hat{\Pi}\} \doteq \hat{\psi}\hat{\Pi} + \hat{\Pi}\hat{\psi} \tag{33}$$

The Dirac Hamiltonian assumes the form

$$\widehat{H}_{\mathrm{D}} = \hat{\bar{\psi}}\hat{\psi} = (\hat{b}^{+}\hat{b} + \hat{c}^{+}\hat{c}) \tag{34}$$

Moreover, a new operator may be introduced in the theory as being proportional to the difference of the number of fermions and number of antifermions. This is known as the charge operator and is represented by

$$\widehat{Q} \doteq e\hat{\bar{\psi}}\gamma^{0}\hat{\psi} = e(\hat{b}^{+}\hat{b} - \hat{c}^{+}\hat{c}) \tag{35}$$

where e is the electron charge. Proceeding in a way similar to the previous section, the conjugate momentum for Dirac c-QFT may be defined as

$$\hat{\Pi}^{\alpha} = \mathrm{i}\hat{\psi}^{+\alpha} \doteq \mathrm{i}\frac{\partial^{\alpha}}{\partial|\psi|^{\alpha}} = \mathrm{i}D^{\alpha} \tag{36}$$

By analogy with (14), the corresponding momentum anti-commutator reads

$$\{\hat{\psi}, \hat{\Pi}^{\alpha}\} = \mathrm{i}\alpha\hat{\Pi}^{(\alpha-1)} \tag{37}$$

Assuming that fermions and antifermions contribute equally to (32), we derive

$$\{\hat{b}^{\alpha}, \hat{b}^{+\alpha}\} = \{\hat{c}^{\alpha}, \hat{c}^{+\alpha}\} = \frac{1}{2}\left[\alpha\widehat{\Pi}^{(\alpha-1)} + 1\right] \tag{38}$$

A logical way to proceed from here is by writing down the anti-commutation relations involving the number and creation operators for fermions and anti-fermions. Retracing the sequence of steps (19)–(21), we arrive at the equation

$$D^{(\alpha-1)}\hat{b}^{+\alpha}|0\rangle = \eta^{(\alpha-1)}\hat{b}^{+\alpha}|0\rangle \tag{39}$$

whose eigenvalues $\eta^{(\alpha-1)}$ form a set of positive and fractional numbers. Considering the same arguments that lead to (21), fermion field excitations generated by $\eta^{(\alpha-1)}$ may be also interpreted as "*complexons*". Moreover, it follows from (35) and (38) that $\widehat{Q}$ generates fractional fermion charges. The emergence of complexons and fractional Dirac charges may be seen as a dynamic manifestation of the high-energy regime that do not have a counterpart in conventional QFT.

5. c-QFT as extended Chern–Simons theory

The emergence of fractional quanta and fractional charges in the last two sections suggests a deeper physical connection between c-QFT and Chern–Simons field theory. As it is known, the Chern–Simons theory may be associated with the occurrence of quasi-particles with fractional spin known as anyons [12,22]. The theory asserts that anyons experience long-range phase interaction mediated by a gauge potential a_μ. The Chern–Simons action for the free theory defined on a closed $2+1$ space–time manifold M has the form

$$\widehat{S} \sim \int_M \varepsilon^{\mu\nu\lambda} \hat{a}_\mu \partial_\nu \hat{a}_\lambda \, \mathrm{d}^3 x \tag{40}$$

in which μ, ν, $\lambda = 0$, i; $(i = 1,2)$ and $\varepsilon^{\mu\nu\lambda}$ represents the totally antisymmetric symbol [22]. A remarkable attribute of (40) is that it is invariant under general coordinate transformations in which the field components change as

$$\hat{a}_\mu(x) = \frac{\partial x'^\lambda}{\partial x^\mu} \widehat{a'}_\lambda(x') \tag{41}$$

As a result, the Chern–Simons action (41) is independent of the geometric structure of the underlying space–time manifold encoded in the metric tensor $g_{\mu\nu}(x)$. It depends only on the topological content of M. It is for this reason that (40) is said to describe a *topological field theory* [22]. Using the language of differential forms, the Chern–Simons term can be expressed as

$$\varepsilon^{\mu\nu\lambda} \hat{a}_\mu \partial_\nu \hat{a}_\lambda \rightarrow \varepsilon \hat{a} \, \mathrm{d}\hat{a} \tag{42}$$

In a formulation involving fractal operators and up to several multiplication constants, (42) may be generalized to

$$\hat{a} \, \mathrm{d}\hat{a} \rightarrow D_x^\delta \hat{a} D_x^\eta \hat{a} \tag{43}$$

The action (40) is accordingly upgraded to

$$\widehat{S}^{\delta,\eta} \sim \int_M D_x^\delta \hat{a} D_x^\eta \hat{a} \, \mathrm{d}^3 x \tag{44}$$

Here, D_x^δ and D_x^η are differential operators with respect to x and δ, η are fractional exponents. We recover the conventional Chern–Simons theory in the limit $\delta = 0$, $\eta = 1$. Under these circumstances, Chern–Simons theory may be regarded as a subset of c-QFT. Taking δ, η to be locally defined $(\delta \doteq \delta(x); \ \ \eta \doteq \eta(x))$ and viewing them as new degrees of freedom, *one may choose to constrain these exponents such as to guarantee invariance of (44) under general coordinate transformations*. A distinctive property of $\delta(x)$, $\eta(x)$ is that they are linearly related to the fractal dimension of the manifold M and thus represent a measure of its underlying topology [13,16]. Following this line of reasoning, we may establish a close link between topological aspects of the Chern–Simons field theory, on one hand, and c-QFT, on the other.[2]

6. Classical limit of c-QFT and general relativity

Expanding on this viewpoint, we may argue that, under the most general circumstances, all exponents entering the generalized Lagrangian (6) are not arbitrary inputs but dynamical parameters that may be fixed by any set of symmetry requirements imposed on the generalized action

$$S_G^{\alpha_1(x),\alpha_2(x),\ldots,\alpha_n(x)} \doteq \int_M L_G^{\alpha_1(x),\alpha_2(x)\ldots,\alpha_n(x)} \, \mathrm{d}^4 x \tag{45}$$

In particular, the generalized action for free fermions may be made invariant with respect to local gauge transformations *without adding gauge fields to the theory*. Setting the right functional constraints on exponents rather than relying on additional dynamical fields to satisfy local gauge symmetry is the key argument of [18].

Given the topological roots of exponent α and its dynamical role in the development of c-QFT, it is of interest to explore how fractal attributes encoded by α may be mapped onto the underlying metric of M. To this end, consider the Lagrangian of the classical scalar field theory (9) in four-dimensional space–time

[2] In this context it is instructive to recall that the Chern–Simons model provides a theoretical framework for explaining the fractional quantum Hall effect, a well known phenomenon in condensed matter physics [12,22].

$$L = (\partial\varphi/\partial t)^2 - \sum_k (\partial\varphi/\partial x^k)^2 - m^2\varphi^2 \tag{46}$$

The generalized Lagrangian built from (46) assumes the form

$$L_G^\alpha = (\partial\varphi/\partial t)^2 - \sum_k D_{x_k}^{2\alpha}\varphi - m^2\varphi^2 \tag{47}$$

An alternate expression for the fractional derivative $D_{x_k}^\alpha \varphi$ is [20]

$$D_{x_k}^\alpha \varphi(x_k) = \frac{\varphi(0)x_k^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)}\int_0^{x_k}(x_k-\xi)^{-\alpha}\frac{\partial\varphi}{\partial x_k}(\xi)\,\mathrm{d}\xi \tag{48}$$

Let the coordinate of point ξ_k $(0 < \xi_k \leqslant x_k)$ be defined as an arbitrary fraction of the endpoint coordinate $x_k \ll 1$, that is, $\xi_k \doteq s x_k$, with $s \leqslant 1$. Assuming, for the sake of simplicity, that the first term in (48) is negligible in comparison with the second term, we derive the following approximation

$$D_{x_k}^\alpha \varphi\left(\frac{\xi_k}{s}\right) \approx \frac{(1-s)^{-\alpha}}{s^{1-\alpha}\Gamma(1-\alpha)}\xi_k^{(1-\alpha)}\frac{\partial\varphi}{\partial x_k}(\xi_k) \tag{49}$$

and

$$D_{x_k}^{2\alpha}\varphi(\xi_k) \approx g_{ik}^\alpha(\xi_k)\left[\frac{\partial\varphi}{\partial x_i}(\xi_i)\right]\left[\frac{\partial\varphi}{\partial x^k}(\xi_k)\right] \tag{50}$$

Up to a product of multiplicative factors independent of ξ_k, the metric $g_{ik}^\alpha(\xi_k)$ is given by

$$g_{ik}^\alpha(\xi_k) \sim \eta_{ik}\xi_k^{2(1-\alpha)} \tag{51}$$

where η_{ik} is the Minkowski metric of special relativity.

On account of (50) and (51), we are led to conclude that the classical limit of c-QFT for free scalar bosons may be formally interpreted as a classical field theory *in curved space–time*. It can be seen that (51) reduces to the metric of special relativity when the fractal topology of space–time makes the transition to a smooth topology, i.e. in the classical limit $\alpha \to 1$. The equivalent metric (51) is subject to the constraint briefly discussed in Appendix D.

7. Concluding remarks

We have laid out the groundwork for complex-quantum field theory using the methodology of fractal differential and integral operators. Our framework has been developed with emphasis on canonical quantization and has led to the following conclusions: (i) the Fock space of c-QFT includes *fractional numbers* of particles and antiparticles per state, (ii) c-QFT represents a generalization of topological field theory and (iii) classical limit of c-QFT is equivalent to field theory in *curved space–time*. The first finding is rooted in the non-commutative nature of space–time in the TeV regime and provides a field-theoretic justification for the transfinite discretization procedure of El Naschie's $\varepsilon^{(\infty)}$ model. The second and third findings suggest the dynamic unification of boson and fermion fields as particles with fractional spin, as well as the close connection between spin and topological attributes of space-time beyond the conventional physics of the standard model.

Future research efforts may be directed towards developing the complexon algebra, understanding the connection between fractional spin and c-QFT and formulating predictions that can be tracked and tested at the Large Hadron Collider and next-generation accelerators. In particular, it is of great interest to explore if c-QFT gives us clues about the particle content of dark matter and its relationship to the TeV physics.

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Appendix A

Fractional derivative of order $0 < \alpha < 1$ described by (5) may be alternatively expressed as a convolution, i.e.

$$D_>^\alpha f(q) \doteq f(q) * \Lambda_\alpha^+(q) \tag{A1}$$

where

$$\Lambda_\alpha^+(q) \doteq \frac{q^{-\alpha}}{\Gamma(1-\alpha)} \quad (q>0) \tag{A2}$$

Similarly we can introduce

$$D_<^\alpha f(q) \doteq f(q) * \Lambda_\alpha^-(q) \tag{A3}$$

with

$$\Lambda_\alpha^-(q) = \Lambda_\alpha^+(-q) \doteq \frac{q^{-\alpha}}{\Gamma(1-\alpha)} \quad (q<0) \tag{A4}$$

Let

$$\hat{p}^\alpha \psi(q) = -\mathrm{i} D_>^\alpha \psi(q) \tag{A5}$$

stand for the fractional momentum operator working on the wavefunction $\psi(q)$. In accordance with the standard formalism of quantum mechanics, its average is given by

$$\langle \hat{p}^\alpha \rangle \doteq \int_{-\infty}^{\infty} \psi^*(q)(-\mathrm{i}D_>^\alpha)\psi(q)\,\mathrm{d}q \tag{A6}$$

To keep the notation simple, we omit throughout the text the subscript ">". Hence we set

$$D_>^\alpha \equiv D^\alpha \tag{A7}$$

Fractional momentum is a linear operator since it satisfies

$$\begin{aligned} &\hat{p}^\alpha \psi_1 = \psi_2 \\ &\hat{p}^\alpha(\psi_1+\psi_2) = \hat{p}^\alpha\psi_1 + \hat{p}^\alpha\psi_2 \\ &C\hat{p}^\alpha\psi = \hat{p}^\alpha(C\psi) \end{aligned} \tag{A8}$$

where C is an arbitrary constant. The integration by parts formula [21]

$$\int_{-\infty}^{\infty} \psi^*(q)(-\mathrm{i}D_>^\alpha)\psi(q)\,\mathrm{d}q = \int_{-\infty}^{\infty} \psi(q)(-\mathrm{i}D_<^\alpha)\psi^*(q)\,\mathrm{d}q \tag{A9}$$

implies that the fractional momentum operator is hermitean if (and only if) we adopt the following definition

$$-\mathrm{i}D_<^\alpha \doteq (-\mathrm{i}D_>^\alpha)^* = \mathrm{i}(D_>^\alpha)^* \tag{A10}$$

Appendix B. Derivation of the commutator $[\hat{\varphi}, \hat{\pi}^\alpha] = \mathrm{i}\alpha\hat{\pi}^{(\alpha-1)}$

Start from (11) and the formal commutator definition

$$[\hat{\varphi}, \hat{\pi}^\alpha]|\varphi\rangle = (-\mathrm{i})[\hat{\varphi}D^\alpha|\varphi\rangle - D^\alpha(\hat{\varphi}\cdot|\varphi\rangle)] \tag{B1}$$

and apply the generalized Leibniz rule [20]

$$D^\alpha(\hat{\varphi}\cdot|\varphi\rangle) = \sum_{m=0}^{\infty}\binom{\alpha}{m} D^m\varphi D^{\alpha-m}|\varphi\rangle = \varphi D^\alpha|\varphi\rangle + \binom{\alpha}{1} D^{(\alpha-1)}|\varphi\rangle \tag{B2}$$

in which

$$\binom{\alpha}{m} \doteq \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-m)\Gamma(1+m)} \tag{B3}$$

Therefore

$$\binom{\alpha}{1} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \alpha \tag{B4}$$

From (B1)–(B4) we derive

$$[\hat{\varphi}, \hat{\pi}^\alpha] = \mathrm{i}\alpha\hat{\pi}^{(\alpha-1)} \tag{B5}$$

Appendix C. Fractional eigenvalue equation $\hat{\pi}^{(\alpha-1)}\hat{a}^{+\alpha}|0\rangle = \frac{\lambda}{\alpha}\hat{a}^{+\alpha}|0\rangle$

Consider

$$D^{(\alpha-1)}\hat{a}^{+\alpha}|0\rangle = \lambda^{(\alpha-1)}\hat{a}^{+\alpha}|0\rangle \tag{C1}$$

where

$$\lambda^{(\alpha-1)} = \frac{\mathrm{i}\lambda}{\alpha} \tag{C2}$$

Here, we have employed the notation

$$\hat{a}^{+\alpha} \equiv \hat{a}^{+\alpha}(\varphi) \tag{C3}$$

The general solution of the above fractional eigenvalue equation subject to the boundary condition [20]

$$D^{(\alpha-2)}\hat{a}^{+\alpha} = (\hat{a}^{+\alpha})_1 \quad \text{as } \varphi \to 0 \tag{C4}$$

is represented by

$$\hat{a}^{+\alpha} = (\hat{a})_1^{+\alpha}\varphi^{\alpha-2}E_{\alpha-1,\alpha-1}\left[\lambda^{(\alpha-1)}\varphi^{(\alpha-1)}\right] \tag{C5}$$

in which $E_{\alpha,\beta}(x)$ denotes the Mittag–Lefler function of order α, β. To determine the eigenvalue spectrum $\lambda_n^{(\alpha-1)}$, we use a boundary condition that fixes the behavior of $\hat{a}^{+\alpha}$ and $(\hat{a}^{+\alpha})_1$ as the scalar field φ approaches its upper limit $\varphi \to \varphi_0$, namely

$$\hat{a}^{+\alpha}(\varphi_0)|0\rangle = A^{+\alpha}(\varphi_0)|0\rangle \tag{C6}$$

On the other hand we have, starting from the boundary condition definition (C4),

$$(\hat{a}^{+\alpha})_1(\varphi)|0\rangle = A_1^{+\alpha}(\varphi)|0\rangle \quad \text{as } \varphi \to 0 \tag{C7}$$

where it is assumed that

$$A_1^{+\alpha}(0) = A_1^{+\alpha}(\varphi_0) \tag{C8}$$

This ansatz leads to the following implicit equation for $\lambda_n^{(\alpha-1)}$

$$A^{+\alpha}(\varphi_0) = A_1^{+\alpha}(0)\varphi_0^{\alpha-2}E_{\alpha-1,\alpha-1}\left[\lambda^{(\alpha-1)}\varphi_0^{\alpha-1}\right] \tag{C9}$$

Appendix D

The equivalent metric must transform in a way that maintains invariance of the space–time interval under arbitrary coordinate changes $x \to \bar{x}(x)$. Hence, in general

$$\bar{g}^{\bar{\alpha}}_{\lambda\sigma}(\bar{x})(\mathrm{d}\bar{x}^{\lambda})^{\bar{\alpha}}(\mathrm{d}\bar{x}^{\sigma})^{\bar{\alpha}} = g^{\alpha}_{\mu\nu}(x)(\mathrm{d}x^{\mu})^{\alpha}(\mathrm{d}x^{\nu})^{\alpha} \tag{D1}$$

where $\bar{\alpha}$ labels the exponent corresponding to the reference frame $\bar{x}$. On account of (5), the partial derivative $\partial^{\bar{\alpha}}(x^{\mu})^{\alpha}/\partial(\bar{x}^{\lambda})^{\bar{\alpha}}$ may be defined as

$$\frac{\partial^{\bar{\alpha}}(x^{\mu})^{\alpha}}{\partial(\bar{x}^{\lambda})^{\bar{\alpha}}} = \frac{1}{\Gamma(1-\bar{\alpha})}\int_{-\infty}^{\bar{x}^{\lambda}} \frac{\partial(x^{\mu})^{\alpha}}{\partial\bar{\xi}^{\lambda}} \frac{\mathrm{d}\bar{\xi}^{\lambda}}{(\bar{x}^{\lambda}-\bar{\xi}^{\lambda})^{\bar{\alpha}}} \tag{D2}$$

The formal connection between $\bar{g}^{\bar{\alpha}}_{\lambda\sigma}(\bar{x})$ and $g^{\alpha}_{\mu\nu}(x)$ may be consequently obtained upon replacing (D2) in (D1).

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Critical behavior in continuous dimension, ε^{∞} theory and particle physics

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Abstract

Bringing closure to the host of open questions posed by the current standard model for particle physics (SM) continues to be a major challenge for the theoretical physics community. Despite years of multiple research efforts, a consistent and comprehensive understanding of standard model parameters is missing. Our work suggests that critical dynamics of the renormalization group flow provides valuable insights into most of the unresolved issues surrounding SM. We report that the dynamics of the renormalization group flow and the topological approach of El Naschie's ε^{∞} theory are viewpoints that share a common foundation. The paper concludes with a brief overview of future developments and integration efforts.

1. Introduction and motivation

As of today, predictions inferred from the standard model of elementary particles (SM) – a body of knowledge discovered in the early 1970s – agree with all the experiments that have been conducted to date. Nevertheless, the majority of particle theorists feel that SM is not a complete framework, but rather an "effective field theory" that needs to be amended by new physics at some higher energy scale reaching in the TeV region. The most cited reasons for this belief are as follows: (a) the recent discovery of neutrino oscillations and masses; (b) SM does not include the contribution of gravity and gravitational corrections to both quantum field theory and renormalization group (RG) equations; (c) SM does not fix the large number of free parameters that enter the theory (in particular the spectra of masses, gauge couplings and fermion mixing angles); (d) SM has a gauge hierarchy problem, which requires fine-tuning; (e) SM postulates that the origin of electroweak symmetry breaking is the Higgs mechanism, whose confirmation is sought in future accelerator experiments. The number and physical attributes of the Higgs boson are neither explained by SM nor fixed from first principles; (f) SM does not clarify the origin of its underlying $SU(3) \times SU(2) \times U(1)$ gauge group and why quarks and leptons occur in certain representations of this group; and (g) SM does not explain why the weak interactions are chiral, i.e. sensitive to fermion handedness.

Despite years of research on multiple fronts, there is currently no compelling and universally accepted resolution to these challenges. A large body of proposed extensions of SM exists, each of them attempting to resolve some unsatis-

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factory aspects of the theory while introducing new unknowns. Expanding on a series of novel and unorthodox contributions of recent years [6–8,10–14,16–18,20], the present work asserts that critical dynamics of the RG flow lies at the root of the many unsettled questions regarding SM. Specifically, we argue that the generation structure of SM and its physical parameters stem from either one of the universal period doubling or golden-mean routes to chaos of the RG flow. Moreover, following [18], we re-iterate that critical behavior in continuous space-time dimension d and above the upper energy limit of SM, (a) is the source of the $SU(3) \times SU(2) \times U(1)$ symmetry group; (b) offers a natural unification mechanism of gauge boson and fermion fields, including classical gravitation and (c) offers a plausible explanation for chiral properties of weak interactions.

Since the concept of dimension and transfinite structure of space-time are the backbone of El Naschie's ε^{∞} theory [10–14], the direct consequence of these results is that the nonlinear dynamics view of the critical RG flow and the topological approach of ε^{∞} share a common foundation.

The paper is structured according to the following plan: Section 2 presents a brief account on RG, critical phenomena and the relevance of the concept of dimension in quantum field theory and the ε^{∞} model. Critical behavior in continuous dimension and its implications for the dynamics of the coupling flow are discussed in Section 3. Sections 4 and 5 analyze the evolution of the coupling flow following the period doubling and golden-mean routes to chaos in simple nonlinear maps. The implications of these routes to chaos on the physics of SM and the link to ε^{∞} theory are elaborated upon in Section 5. An outline on future developments and integration efforts is presented in Section 6. Conclusions are summarized in the last section.

2. Renormalization group, critical phenomena and the concept of dimension

The scale of correlations in statistical physics and quantum field theory is known to become unbounded in the vicinity of a critical point. In this region, fluctuations lead to singular thermodynamic behavior characterized by universal critical exponents and scaling functions [1–3]. These power-law singularities bring to light an underlying emerging symmetry associated with critical phenomena, namely the manifest scale invariance of the theory: at the critical point, the physical system has no characteristic scale and the correlation length diverges. RG provides a natural framework for explaining the onset of critical phenomena, the roots of universality and the classification of various systems in terms of universality classes. In the context of RG, the process of integrating out fluctuations and the short-distance degrees of freedom is made systematic. For instance, if there is a single mass scale M in the microscopic theory, RG proceeds by building an effective field theory whose content may be understood as a power expansion in $1/M$. Stated differently, RG is based on the idea that the renormalization technique absorbs all relevant fluctuations above M into a finite number of parameters that define the theory. There are two implicit premises behind this technique: (a) fluctuations have a finite average and (b) renormalization process is carried out at a fixed dimensionality of the space-time background.

Regarding the second premise, a key consequence of RG in both statistical physics and field theory is that universal properties near second-order phase transitions depend strongly on the space-time dimensionality (d). Consider, for instance, the traditional one-component Ising model consisting of an orthogonal lattice of spins experiencing nearest neighbor coupling. It can exhibit an infinite number of multi-critical points in $d = 2$, a critical Wilson–Fisher or a tri-critical point in $d = 3$ and a Gaussian fixed point for $d = 4$ [4]. Percolation, random walks and formation of fractal clusters in systems undergoing second-order phase transitions are also typical examples of processes whose outcome depends on d [1–3]. The relevant literature on statistical physics of phase transitions points out that continuity in the dimensionality of space-time is an essential ingredient for the correct description of critical phenomena. To be specific, extrapolation from $d = 4$ to an infinitesimally lower dimension $\varepsilon = 4 - d$ is the basis for dimensional regularization in field theory and represents one frequent method in the non-perturbative study of RG flow near non-trivial fixed points [1–5]. Recent work on field theories formulated in continuous dimension asserts that a new type of critical behavior develops at a fixed energy scale μ as a result of incremental changes in $\varepsilon = 4 - d$ [4,5].

It is important to emphasize in this context that ε^{∞} theory also relies on the infinite dimensional structure of the space-time manifold. The fundamental premise of ε^{∞} is that space-time may be thought of as a transfinite collection of primary Cantor sets having a random distribution of underlying dimensions [10–14]. Three key dimensions assume the leading role as topological invariants of the theory: the formal infinite dimension $n = \infty$, the expectation value $\langle d \rangle =\sim \langle n \rangle = 4 + \phi^3$, where ϕ represents the golden mean, and the topological dimension $n_t = 4$. Although the contribution of ϕ^3 and transfinite corrections are negligible at low-energy scales, they become an underlying source for emergent behavior in the high-energy sector of field theory. It is in this far-from-equilibrium setting where vacuum fluctuations and the complex topology of the space-time manifold create a rich reservoir for extensive symmetry breaking and pattern formation [10]. *On this basis, one can establish a deep analogy between the dynamics of critical phenomena* *in continuous dimension and the complex topology of El Naschie's ε^{∞} model.* Some of the physical implications of this analogy are briefly discussed in Section 5.

3. Critical phenomena in continuous dimension

It is known that the dependence of the coupling charge on the energy scale is a basic outcome of RG. Following the generic formulation of RG in quantum field theory [6,7,9], the beta-function defines how the coupling charges g_i "flow" with the sliding energy scale μ, that is

$$\beta_i(g_i) \doteq \frac{\mathrm{d}(g_i)}{\mathrm{d}t} \tag{1}$$

where the evolution parameter

$$t \doteq \ln\left(\frac{\mu}{\mu_0}\right) \tag{2}$$

is considered at an arbitrary reference scale μ_0. To fix ideas, let us consider the example of beta-functions that describe the gauge-coupling sector of SM. At the one-loop contribution of the perturbation expansion, one has [21]

$$(4\pi)^2\beta_i(g_i) = b_i g_i^3 \tag{3}$$

Here, g_i, $i = 1,2,3$ are related to the conventional SM gauge couplings via $g_1 = \sqrt{5/3}g'$, $g_2 \equiv g$, $g_3 = g_{\mathrm{QCD}}$ and the triplet of one-loop coefficients is given by $b_1 = \frac{41}{10}$, $b_2 = -\frac{19}{6}$, $b_3 = -7$, respectively. In order to simplify notation, we ignore in what follows the index i and take $\mu_0 = M_Z$, the mass of the Z^0 boson. In the basin of attraction of a critical point t_{c} the field correlation length scales as

$$\xi \approx |t - t_c|^{\nu} \tag{4}$$

Here, critical exponent ν is given by [4,5]

$$\nu^{-1} = -\left.\frac{\partial\beta}{\partial(g)}\right|_{g=g^*} \tag{5}$$

and g^* stands for a fixed point of the beta-function, $\beta(g^*) = 0$.

The continuity of the beta-function with respect to d makes it possible to interpret dimension d as a control parameter in the same way the energy scale acts as a control parameter in RG. According to this philosophy, the sliding scale t and d play an interchangeable role. Assume that d_{c} represents the critical dimension for which g flows into the fixed point g^*. The mass of the underlying field is known to be inversely proportional to the divergent correlation length and identically vanishes at the fixed point

$$M[g^*(d_{\mathrm{c}}), d_{\mathrm{c}}] = 0 \tag{6}$$

In the basin of attraction of g^* the field develops mass according to the power law [4,5]

$$M[g^*(d_{\mathrm{c}}), d_{\mathrm{c}} - d] \approx |d_{\mathrm{c}} - d|^{\nu(d_{\mathrm{c}})} = \varepsilon^{\nu(d_{\mathrm{c}})} \tag{7}$$

where by analogy with (5)

$$\nu^{-1}(d_{\mathrm{c}}) = -\left.\frac{\partial\beta}{\partial(g)}\right|_{g^*(d_{\mathrm{c}})} \tag{8}$$

We are led to conclude that, as the fixed point is asymptotically approached and the continuous space-time dimensionality collapses to $d \to d_{\mathrm{c}} = 1,2,3,4$ the underlying field theory becomes massless, in agreement with the current principles of relativistic quantum field theory [9]. Numerical analysis yields $\nu(d_{\mathrm{c}}) = 0.5$ for $d_{\mathrm{c}} = 1,2,3,4$ which is found to match well the value reported in the literature [4,5].

An important observation is now in order. Following universality arguments related to the onset of chaos in low-dimensional maps, the dimensional control parameter $\varepsilon = |d_{\mathrm{c}} - d|$ is expected to asymptotically approach the critical value $\varepsilon_{\infty} = 0$ according to the geometric progression [24,25]:

$$\varepsilon_n - \varepsilon_{\infty} \approx a_n \cdot \delta^{-n} \tag{9}$$

in which $n \gg 1$ is the index defining the number of iteration steps, δ is a scaling constant that is representative for the class of dynamical maps under consideration and a_n is a coefficient which becomes asymptotically independent of n, that is, $a_{\infty} = a$. Substituting (9) in (7) produces to the mass scaling series

$$\boxed{M_n[g^*(d_c), d_c - d_n] \approx (a_n \delta^{-n})^{1/2}} \tag{10}$$

We may go a step further and state that, given the link between the coupling flow and the evolution of masses and fields in RG [9], similar scaling pattern develops for g_n and the underlying fields of the theory, η_n. We thus expect to obtain, for $n \gg 1$

$$\begin{aligned} &\boxed{g_n - g^*(d_c) \approx \delta^{-\lambda(d_c)n}} \\ &\boxed{\eta_n - \eta^*(d_c) \approx \delta^{-\zeta(d_c)n}} \end{aligned} \tag{11}$$

where $\lambda\ (d_c)$ and $\zeta(d_c)$ represent two additional critical exponents dependent on d_c.

4. Chaotic dynamics of the beta-function

The goal of this section is to use the simplest models that simulate the chaotic dynamics of the beta-function (3) and invoke universality principles to extrapolate results to more realistic settings.

As it is known, a key issue in the study of deterministic dynamical systems is to identify the different types of asymptotic behavior defined by $n \gg 1$ and to understand how this behavior changes under incremental variations of the control parameter. The asymptotic behavior can be, for instance, a steady state, a periodic oscillation, a quasi-periodic or a chaotic motion. There are four traditional routes to chaotic attractors in generic maps: period doubling, intermittency, crises and quasiperiodicity. In what follows, we focus exclusively on: (a) the period doubling cascade corresponding to the Feigenbaum attractor in maps with quadratic maxima and (b) the quasi-periodic route to chaos through irrational winding numbers (the so-called "golden-mean" approach to chaos) [23,24].

4.1. Period doubling transition to chaos

Eq. (3) may be viewed as describing the dynamics of a free over-damped oscillator with cubic interaction [24]:

$$\frac{\partial^2 g}{\partial t^2} + \gamma \frac{\partial g}{\partial t} + \omega^2 g - b g^3 = 0 \tag{12}$$

Here, $\partial^2 g/\partial t^2 \approx \mathrm{O}(\varepsilon)$ denotes the second-order derivative of the coupling with respect to the fictitious time $t = \ln(\mu/M_Z)$, $\gamma = (4\pi)^2$ stands for the damping parameter and $\omega \approx \mathrm{O}(\varepsilon)$ is the fictitious oscillator frequency. A straightforward way to simulate the effect of sustained fluctuations induced by dimensional variation near the fixed points (d_c) is to add to (12) an infinite series of periodic kicks as in

$$\frac{\partial^2 g}{\partial t^2} + \gamma \frac{\partial g}{\partial t} + \omega^2 g - b g^3 = \sum_j \varepsilon \delta(t - jT) \tag{13}$$

Here, T denotes the interval between kicks and $\varepsilon = |d_c - d|$ is their amplitude. Following [19], we introduce the complex variable

$$\lambda = \left(\mathrm{i}g + \frac{1}{\omega}\frac{\partial g}{\partial t}\right)\sqrt{\frac{3}{8}\frac{b}{\omega}\frac{\mathrm{e}^{-\gamma T} - 1}{\gamma}} \tag{14}$$

which transforms (13) into the so-called Ikeda map

$$\lambda_{n+1} = A + B\lambda_n \exp(\mathrm{i}(|\lambda_n|^2 + \psi)) \tag{15}$$

where

$$\begin{aligned} A &= \frac{\varepsilon}{\omega}\sqrt{\frac{3}{8}\frac{b}{\omega}\frac{\mathrm{e}^{-\gamma T} - 1}{\gamma}} \\ B &= \mathrm{e}^{-\gamma T/2} \\ \psi &= \omega T \end{aligned} \tag{16}$$

In what follows, we focus on the effect of sufficiently small kicking intervals $T \approx \mathrm{O}(\varepsilon)$ commensurate with the reference scale $\mu_0 = M_Z$. This particular case corresponds to $B = 1, \psi = \mathrm{O}(\varepsilon)$ and yields a finite absolute value for A since $\omega \approx \mathrm{O}(\varepsilon)$. Furthermore, assuming for the sake of simplicity a strongly over-damped system, we have $\gamma\partial g/\partial t \gg \partial^2 g/\partial t^2$ and $A \gg 1$. Under these circumstances, it can be shown that (15) may be transformed into the following one-dimensional map [24]

$$\sigma_{n+1} = A^2[2\cos(\sigma_n) + 1] \tag{17}$$

upon performing the substitution

$$\sigma = A\left[A + \left(\lambda + \overline{\lambda}^*\right)\right] \tag{18}$$

The map (17) exhibits the well-known path to the Feigenbaum attractor via universal period doubling bifurcations in maps with quadratic maxima. The corresponding sequence of control parameters A^2 (and, implicitly, the sequence of dimensional parameters ε^2) converges to the Feigenbaum constant $\delta_2 = 4.669\ldots$, i.e.

$$\varepsilon_n \approx \delta_2^{-n/2} \tag{19}$$

A more accurate representation is obtained by accounting for the contribution of higher order corrections to the Feigenbaum scaling [22]

$$\varepsilon_n \approx \left(\sum_k \Delta_k \delta_k^{-n}\right)^{1/2} \tag{20}$$

in which Δ_k and δ_k stand for the generic expansion coefficient and Feigenbaum constant of order k.

4.2. Golden-mean transition to chaos

To link (3) with the onset of chaos through instability of quasi-periodic orbits, we assume below that: (a) the cubic term may be supplemented with higher order contributions that simulate the action of periodic fluctuations, (b) the gauge-coupling equation is perturbed by a term proportional to the dimensional control parameter ε. Hence, we proceed by extrapolating (3) to

$$(4\pi)^2\beta(g) = b\sin(g) + \varepsilon \tag{21}$$

which may be cast in the form of a sine circle map, that is

$$g_{n+1} = \varepsilon + g_n + K\sin(g_n) \tag{22}$$

and where the overall fluctuation amplitude is

$$K = \frac{b}{(4\pi)^2} \tag{23}$$

The sequence of dimensional parameters obeys the universal scaling

$$\varepsilon_n(K) \approx c\overline{\delta}^{-n} \tag{24}$$

in which c is a scaling coefficient and the so-called Kadanoff–Feigenbaum–Shenker (KFS) constant depends on the golden-mean ϕ according to [23,24]

$$\overline{\delta} = \phi^{-2} = \left(\frac{\sqrt{5}-1}{2}\right)^{-2} \tag{25}$$

By analogy with the period doubling scenario, a better approximation of (24) may be obtained through the use of higher order scaling corrections, that is

$$\varepsilon_n(K) \approx c\left(\sum_k \overline{\Delta}_k \overline{\delta}_k^{-n}\right)^{1/2} \tag{26}$$

where $\overline{\Delta}_k$ and $\overline{\delta}_k$ denote the generic expansion coefficient and KFS constant of order k.

5. Universal scaling of SM parameters

5.1. The period doubling scenario

For period-doubling bifurcations we take $n = 2^m$, with $m > 1$. Replacing in (10) yields the following mass series:

$$M_m \approx \sqrt{a_{2^m}} \cdot \delta^{-\frac{2^m}{2}\cdot\frac{1}{2}} = \sqrt{a_{2^m}} \cdot \delta^{-2^{m-2}} \tag{27}$$

The ratio of two arbitrary masses is therefore

$$\frac{M_l}{M_m} \approx \sqrt{\frac{a_{2^l}}{a_{2^m}} \cdot \frac{\delta^{-2^{l-2}}}{\delta^{-2^{m-2}}}} \tag{28}$$

where $\lim \frac{a_l}{a_m} = 1$ as $l, m \to \infty$. Thus, for two consecutive terms in the mass series and ignoring higher order scaling corrections,

$$\boxed{\frac{M_l}{M_{l+1}} \approx \sqrt{\frac{a_{2^l}}{a_{2^{l+1}}}} \cdot \delta^{2^{l-2}} = \sqrt{a_{2^l,2^{l+1}}} \cdot \delta^{2^{l-2}}} \tag{29}$$

It is important to emphasize that (29) provides only a first-order approximation considering that (a) no higher order corrections are accounted for, (b) (29) is less accurate if the iteration index is not large enough, that is, if $l \approx O(1)$, and (c) there is a fair amount of uncertainty involved in determining the quark mass spectrum [15].

Numerical results derived from (29) are displayed in the table below. This table contains a side-by-side comparison of estimated versus actual mass ratios for charged leptons and quarks and a similar comparison of gauge-coupling ratios. All masses and couplings are evaluated at the energy scale given by the top quark mass. Quark masses are averaged using the most recent reports issued by the Particle Data Group [15]. Specifically, $m_u = 2.12$ MeV; $m_d = 4.22$ MeV; $m_s = 80.9$ MeV; $m_c = 630$ MeV; $m_b = 2847$ MeV; $m_t = 170{,}800$ MeV (see Table 1).

The scaling sequence of charged leptons and quarks may be graphically summarized with the help of the following diagrams:

$$\overbrace{d \quad s}^{\delta^2} \qquad \overbrace{s \quad b}^{\delta^2}$$

$$\overbrace{u \quad c}^{\delta^4} \qquad \overbrace{c \quad t}^{\delta^4}$$

$$\overbrace{e \quad \mu}^{\delta^4} \qquad \overbrace{\mu \quad \tau}^{\delta^2}$$

$$\overbrace{\alpha_{EM}^2 \quad \alpha_W^2}^{\delta^2} \qquad \overbrace{\alpha_W^2 \quad \alpha_s^2}^{\delta^2}$$

Based on the above scheme, one may infer that one of the possible patterns for neutrino masses is given by:

$$\overbrace{\nu_e \quad \nu_\mu}^{\delta^{16}} \qquad \overbrace{\nu_\mu \quad \nu_\tau}^{\delta^4}$$

Table 1
Fermion masses and coupling ratios

Scaling ratio	2^{l-2}	Experimental value	Estimated value
m_u/m_c	4	3.365×10^{-3}	2.104×10^{-3}
m_c/m_t	4	3.689×10^{-3}	2.104×10^{-3}
m_d/m_s	2	0.052	0.046
m_s/m_b	2	0.028	0.046
m_e/m_μ	4	4.745×10^{-3}	2.104×10^{-3}
m_μ/m_τ	2	0.061	0.046
$(\alpha_{EM}/\alpha_W)^2$	2	0.053	0.046
$(\alpha_{EM}/\alpha_s)^2$	4	4.034×10^{-3}	2.104×10^{-3}

It is also instructive to note that quarks and charged leptons follow a different period doubling pattern. To this end, let us organize the charged lepton and quark masses in a collection of triplets, that is

$$M_l \doteq [m_e \quad m_\mu \quad m_\tau], \quad M_q \doteq \begin{bmatrix} m_u & m_d \\ m_c & m_s \\ m_t & m_b \end{bmatrix} \tag{30}$$

It can be seen that the mass scaling for adjacent quarks stays constant within either one of the triplets (u, c, t) or (d, s, b), whereas the mass scaling for charged leptons varies as a geometric series in δ_2^2 within the triplet (e, μ, τ). This finding points out toward a symmetry breaking mechanism that segregates lepton and quark phases in the process of "cooling" from the far ultraviolet region of field theory to the low-energy region of SM.

5.2. The golden-mean scenario and ε^∞ theory

In this case we start by substituting (25) in (10). Ignoring the higher order scaling contributions, the generic term of the mass series becomes

$$\boxed{M_n \approx a_n \cdot \phi^n} \tag{31}$$

The above formula opens the door for a straightforward connection to ε^∞ theory, where the golden mean plays a key dynamic role [10–14]. A couple of important observations need to be brought up here:

(a) the exponent $\nu(d_c) = 1/2$ introduced in Section 3 describes critical behavior in continuous dimension whereby $d_c = 1, 2, 3, 4$. This type of behavior occurs in a space-time endowed with an ordinary topological structure. By contrast, the complex topology of ε^∞ requires a new critical exponent, that is, $\nu_\infty(d_c) \neq 1/2$;
(b) higher order scaling corrections alluded to in Section 4 are replaced in ε^∞ theory by the so-called transfinite corrections. Among the most important transfinite corrections we mention

$$\begin{aligned} \phi^3 &= \langle d_c \rangle - 4 \\ k &= \phi^3(1 - \phi^3) \\ k_0 &= \phi^5(1 - \phi^5) \end{aligned} \tag{32}$$

It is apparent from (32) that the role of the dimensional parameter $\varepsilon = 4 - d$ is played in ε^∞ by ϕ^3. From the above observations, it follows that the scaling relations (10) and (31), adapted to ε^∞ theory and taken in the absence of transfinite corrections, assumes the form

$$\boxed{M_n[g^*(d_c), \phi^3] \approx (\phi^3)^{-n \cdot \nu_\infty(d_c)}} \tag{33}$$

which is consistent with the treatment developed in [10–14].

6. Future developments

The ideas presented in this work strongly support the conjecture that the nonlinear dynamics view of the critical RG flow and the topological approach of ε^∞ share a common foundation. Bringing into the picture fractional dynamics may lead to a possible expansion of this conjecture. As shown in [18], the description of complex dynamics in the TeV regime of field theory warrants the transition from ordinary calculus on smooth manifolds to fractional differentiation and integration. This transition has important implications regarding phenomena that are anticipated beyond the energy range of SM. In particular, it is argued in [18] that:

(a) fractional dynamics in Minkowski space-time is equivalent to field theory in curved space-time. This result points out to a natural integration of gravity in the far ultraviolet region of standard field theory;
(b) the $SU(3) \times SU(2) \times U(1)$ gauge group of SM is rooted in the concept of continuous dimension;
(c) fractional dynamics is the underlying source of parity non-conservation in processes involving the $SU(2)$ group. As such, fractional dynamics offers a plausible explanation for the inherent chirality of weak interactions.

7. Summary and conclusions

We have developed arguments suggesting that critical dynamics of the RG flow lies at the root of many unsettled questions surrounding SM. Specifically, it was argued that the generation structure of SM and its physical parameters stem from either one of the universal period doubling or golden-mean routes to chaos of the RG flow. Following this line of reasoning, one may establish a deep analogy between the dynamics of critical phenomena in continuous dimension and the complex topology of El Naschie's ε^{∞} model. A brief discussion on how using fractional dynamics may be able to expand our findings has been presented.

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Fractional dynamics and collider phenomenology

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ABSTRACT

Both theory and experiments strongly suggest that new phenomena await discovery above the energy range of the standard model for particle physics (SM). In this brief report we argue that a correct description of physics in the TeV energy region needs to account for the inherent randomness induced by short-distance fluctuations. The existence of "unparticles", alleged to emerge at the next-generation colliders, is motivated by a dynamic setting that is out-of-equilibrium and able to sustain a rich spectrum of complex phenomena.

1. Introduction

Quantum Field Theory (QFT) is a framework whose methods and ideas have found successful applications in many domains, from particle physics and condensed matter to cosmology, statistical physics and critical phenomena [1,2]. As a fundamental synthesis of quantum mechanics and special relativity, QFT forms the foundation for SM, a body of knowledge that describes the behavior of all known particles and their interactions except gravity. Feynman diagrams are well-established tools for computing transition amplitudes in QFT [1,2]. As particle physics enters the era of high-energy experiments at the Large Hadron Collider (LHC) and International Linear Collider (ILC), one is compelled to ask the following question: How reliable is the apparatus of perturbation theory in the Terascale sector of field theory? To answer this question, it is important to properly define the domain of validity for the path integral (PI) formalism of QFT and the technique of Feynman diagrams. In particular,

(a) The PI formalism is often used in conjunction with so-called "effective field theories" (EFT). EFT are based on the explicit hypothesis that microscopic fields (quantum corrections contributed by heavy excitations) can be coarse-grained and absorbed into a re-definition of the coupling coefficients defining the Lagrangian [3]. This conjecture assumes that microscopic fields are stable and can be effectively shielded from interfering with macroscopic fields. However, overlap continues to exist in the so-called crossover region where fluctuations cannot be fully suppressed [4].
(b) Quantum processes maintain coherence. This ansatz fails in the presence of fast fluctuations that rapidly decohere the system and drive the transition from "quantum" to the "classical" behavior [5].
(c) Evolution is assumed to be unitary, regular, Markovian and described by analytic functions. According to [6], Hamiltonian systems are carriers of chaos. The phase space of an arbitrary Hamiltonian system contains regions where

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motion occurs with a mixing of trajectories. In this instance, the hypothesis of regular evolution and "smooth" trajectories breaks down.

(d) Compliance with special relativity demands that particle processes are strictly local. But the "locality" ansatz is bound to fail near second order phase transitions following the manifest loss of scale associated with critical phenomena. Critical behavior involves cooperative phenomena that evolve on vastly different length scales *while still remaining compliant with relativity*. In this instance, the concept of "locality" cannot be separated from the concept of observation scale: self-similarity enables one to map a non-local process into a local one by an appropriate scale transformation.

It is our view that all these arguments call for a paradigm shift in how field theory is approached beyond SM. A natural question is then: What is the best way to initiate this change of perspective? Owing it to the significant progress in this field, we believe that a promising avenue is the *complex dynamics of non-linear systems*. Pattern formation and self-organized criticality are typical examples of phenomena that display complex behavior [7,8]. Recent years have taught us that complex phenomena seem to show "universality" across vastly different energy regions. Collective behavior is prone to develop in *non-linear* systems that are *open* to environmental or internal fluctuations. Since QFT is essentially based on non-linear gauge models and its short-distance regime describes phenomena that unfold under large perturbations in momentum, it is reasonable to assume that complexity will play a key role in explaining upcoming experiments at LHC, ILC and next-generation accelerators [8–10]. By the same token, analytic tools offered by stochastic dynamics and non-equilibrium statistical physics will most likely be of great utility to this undertaking [22,23].

Recently, the possibility of a scale-invariant hidden sector of particle physics extending beyond SM has attracted a lot of attention [11–15]. A strange consequence of this hypothesis is the emergence of a continuous spectrum of massless fields having non-integral scaling dimensions called "unparticles". Drawing from arguments pertaining to the behavior of Renormalization Group in the presence of random fluctuations [16–18], we suggest herein that the would-be "unparticles" arise due to a dynamic setting that is manifestly stochastic and out-of-equilibrium. It is also suggested that this picture enables a natural explanation for breaking of space-time symmetries in weak interactions. The violation of space-time symmetries has recently been identified as a promising candidate signal for physics beyond SM [19].

We caution that our study has an introductory nature. As such, it does not claim to be fully rigorous or comprehensive. Independent research work is needed to confirm, expand or refute these preliminary findings.

2. Effective field theory and Terascale physics

Following [11,12], we begin with the hypothesis that there is a hidden sector lying beyond SM whose existence is likely to be uncovered at LHC, ILC or future accelerators. To streamline the derivation, we use the EFT prescription [3] and model this sector using a single light field operator $O(\mu)$ in interaction with a single heavy state that emerges in the deep UV region ($\Lambda \gg \Lambda_{SM}$). Here, $\Lambda_{SM} = O(G_F^{-1/2}) \approx 300$ GeV stands for the uppermost bound of SM corresponding to the weak interaction scale. The EFT is then defined by the Lagrangian

$$L_{EFT}(\mu) = c(\mu, \Lambda)O(\mu) = \frac{c_0(\mu)}{\Lambda^{d_0-4}} O(\mu) \tag{1}$$

Here, μ is the sliding scale and d_0 the mass dimension of operator $O(\mu)$

$$[O(\mu)] = \mu^{d_0} \tag{2}$$

Lagrangian (1) contains only the light field operator and the effect of the heavy field is encoded in the coupling constant $c(\mu, \Lambda)$. Our aim is to study the behavior of the theory near its infrared fixed point $\mu_{IR} \approx \Lambda_{SM}$.

According to [16], the light field operator acts as a random object in momentum space. Without any loss of generality, let us define the coarse-grained operator

$$O_R(\mu) = \frac{1}{K} \int O(\eta) W(\mu - \eta) d\eta \tag{3}$$

in which μ stands for the sliding scale and K is normalization constant. The kernel function $W(\mu)$ is linearly related to the coarse-grained probability density of locating a specific value in momentum space $p(O(\mu), c_0(\mu_0))$. It can be shown that the asymptotic form of the coarse-grained probability density near the IR point is given by [16]

$$\lim_{\mu \to \Lambda_{SM}} O(\mu), c_0(\mu_0)] \approx \mu^{-\delta} F\left[\frac{O(\mu_0)}{\mu_0^{d_0}} \left(\frac{\mu}{\mu_0}\right)^{-\delta}; c_0^*\right] \tag{4}$$

where

$$\delta = d_0 + \frac{1}{2}\gamma(c_0^*) \tag{5}$$

Here, the theory is assumed to be massless for simplicity, c_0^* is a fixed point of $c_0(\mu)$, μ_0 denotes an arbitrary reference scale and $\gamma(\cdots)$ represents the so-called anomalous dimension. This universal result indicates that the large scale asymptotic form of the coarse-grained probability density represents a non-trivial power of the sliding scale times a certain dimensionless function $F[\cdots]$. Replacing in (3) yields

$$\lim_{\mu\to\Lambda_{\mathrm{SM}}} O_{\mathrm{R}}(\mu) \propto \int O(\eta)(\mu-\eta)^{-\delta} F\left[\frac{O(\mu_0)}{\mu_0^{d_0}}\left(\frac{\mu-\eta}{\mu_0}\right)^{-\delta}\right] \mathrm{d}\eta \tag{6}$$

Since there is no restriction regarding the choice of μ_0, it is convenient to assume

$$|\mu-\eta| \ll \mu_0 \tag{7}$$

On account of (7), a reasonable approximation of (6) can be presented as

$$\lim_{\mu\to\Lambda_{\mathrm{SM}}} O_{\mathrm{R}}(\mu) \propto \int (\mu-\eta)^{-\delta} \frac{\partial\Omega_1}{\partial\eta} \mathrm{d}\eta + \int (\mu-\eta)^{1-\delta} \frac{\partial\Omega_2}{\partial\eta} \mathrm{d}\eta \tag{8}$$

where

$$O(\eta) \doteq \frac{\partial\Omega_1(\eta)}{\partial\eta}$$
$$O(\eta)\frac{\partial F(\eta)}{\partial\eta}\bigg|_{\mu=\eta} \doteq \frac{\partial\Omega_2(\eta)}{\partial\eta} \tag{9}$$

The next section makes connection to fractional differential operators, fractional dynamics and their relevance to (8).

3. Fractional dynamics: a brief overview

As pointed out in the Introduction, a key assumption of EFT is that microscopic fields are stable and can be completely decoupled from macroscopic fields. Probing matter in the near or deep Terascale sector will likely violate this assumption. In the so-called crossover region, the unavoidable action of un-damped quantum corrections creates a mixing of microscopic and macroscopic fields [4,9]. As a result, instabilities can develop on long time-scales and the macroscopic description of phenomena in terms of conventional differential operators breaks down. This is, in essence, the main argument for using fractional operators in the far TeV region of field theory, for the passage from ordinary to fractional dynamics and non-extensive statistical physics [27,28]. The framework of fractional dynamics offers a reliable tool for the study of far-from equilibrium processes that display scale-invariant properties, dissipation and long-range correlations. It plays nowadays an increasingly important role in many branches of engineering, science and applied mathematics [21,24,26,27] and included references].

Returning to (8) and using the explicit expression of differential operator from fractional calculus, we arrive at

$$\lim_{\mu\to\Lambda_{\mathrm{SM}}} O_{\mathrm{R}}(\mu) \propto D_{\mathrm{C}}^{\delta}\Omega_1(\mu) + D_{\mathrm{C}}^{\delta-1}\Omega_2(\mu) \tag{10}$$

where the Caputo derivative of order α is defined by [20,25]

$$D_{\mathrm{C}}^{\alpha} f(x) \doteq \frac{1}{\Gamma(1-\alpha)} \int (x-\tau)^{-\alpha} \left(\frac{\mathrm{d}f(\tau)}{\mathrm{d}\tau}\right) \mathrm{d}\tau \tag{11}$$

This result confirms that, *near and above* the weak interaction scale Λ_{SM}, conventional differential operators need to be replaced by fractional operators. Our finding is consistent with [16], where it is argued that Renormalization Group in the presence of random fluctuations and interactions describes fractional Brownian motion and complex behavior. We also direct the reader to [9], in which a similar motivation is articulated in greater detail.

4. On the domain of validity of fractional dynamics

One important remark is now in order. It is known that the requirement of unitary evolution represents a fundamental postulate of both quantum mechanics and QFT.Quantum physics describes equilibrium dynamical processes which are unaffected by unitary transformations. The corresponding operators are known to preserve transition probabilities among various states and satisfy a *group property* [1,2].

The weak interaction scale Λ_{SM} is the highest energy threshold probed with the current accelerator technology. There are grounds to suspect that unitary evolution postulated by QFT no longer holds *near or above* Λ_{SM}. Here, quantum processes are expected to evolve in a highly unstable, "noisy" environment and are prone to migrate outside equilibrium [9,10,21,24]. As previously pointed out, it is believed that a correct account of this dynamic regime requires use of fractional operators and fractional dynamics. It is important to understand that, since fractional operators are non-unitary and obey only a semi-group property [25], they *cannot describe the physics of* SM which, by definition, unfolds below Λ_{SM}.

5. Concluding remarks

There are two important consequences that can be drawn from our model:

(1) We have suggested in [9] that the onset of fractional dynamics leads to the emergence of non-integer numbers of particles and antiparticles. The approach developed here reinforces this conclusion: unusual states of matter, if they exist, are directly related to complex dynamics induced by Terascale fluctuations. This is in contrast to [11,12] where "unparticles" emerge from the action of a hidden sector of particle physics that lies beyond SM.

(2) Fractional operators defined on space-time have a built-in asymmetry to the inversion of coordinates. This property enables a natural explanation for breaking of parity and time symmetries in weak interactions [21,24]. The origin of these two symmetry violations is currently an unsettled issue of SM [29].

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Fractional dynamics and the Standard Model for particle physics

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Abstract

Fractional dynamics is an attractive framework for understanding the complex phenomena that are likely to emerge beyond the energy range of the Standard Model for particle physics (SM). Using fractional dynamics and complex-scalar field theory as a baseline, our work explores how physics on the high-energy scale may help solve some of the open questions surrounding SM. Predictions are shown to be consistent with experimental results.

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1. Introduction and motivation

As of 2006, predictions derived from the Standard Model of elementary particles (SM) – a body of knowledge discovered in the early 1970s – agrees with all the experiments that have been conducted to date. Nevertheless, the majority of particle theorists feel that SM is not a complete framework, but rather an "effective field theory" that needs to be extended by new physics at some higher energy scale reaching in the TeV region. The most cited reasons for this belief are: (a) the recent discovery of neutrino oscillations and masses; (b) SM does not include the contribution of gravity and gravitational corrections to both quantum field theory and renormalization group (RG) equations; (c) SM does not fix the large number of free parameters that enter the theory (in particular the spectra of masses, gauge couplings and fermion mixing angles); (d) SM has a gauge hierarchy problem, which requires fine-tuning; (e) SM postulates that the origin of electroweak symmetry breaking is the Higgs mechanism, whose confirmation is sought in future accelerator experiments. The number and physical attributes of the Higgs boson are neither explained by SM nor fixed from first principles, (f) SM does not clarify the origin of its underlying $SU(3) \times SU(2) \times U(1)$ gauge group and why quarks and leptons occur in certain representations of this group, (g) SM does not explain why the weak interactions

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are chiral, that is, why only fermions with one handedness experience the force transmitted by the triplet of massive vector bosons W^{+}, W^{-}, Z^{0}.

Despite years of research on multiple fronts, there is currently no compelling and universally accepted resolution to the above-mentioned challenges. A large body of proposed extensions of SM exists, each of them attempting to resolve some unsatisfactory aspects of the theory while introducing new unknowns. Expanding on a series of recent contributions centered on RG, non-linear dynamics, chaos and fractal geometry [3,4,6–10,12–14,17–19], our work explores how the physics on the TeV regime may shed light onto some of the open questions surrounding SM.

The paper is organized as follows: Section 2 surveys the motivation for fractional dynamics in the far ultraviolet region of field theory. The principle of local scale invariance is briefly introduced in Section 4. Fractional dynamics of a "toy" model based on complex scalar fields is analyzed in Section 5. Sections 6–8 discuss how critical behavior in continuous dimension acts as source of massive field theories and makes connection to SM data. Concluding remarks are presented in the last section. We emphasize from the outset the introductory nature of our work. As such, its content is not aimed to be either entirely rigorous or formally complete. Independent research efforts are required to confirm, develop or disprove these preliminary results.

2. Fractional dynamics and the far ultraviolet region of field theory

It is generally believed that quantum field theory breaks down near the so-called Cohen–Kaplan threshold of ~100 TeV as a result of exposure to large vacuum fluctuations and strong-gravitational effects. No convenient redefinition of observables is capable of turning off the dynamic contribution of these effects. For instance, it is known that the zero-point vacuum energy diverges quadratically in the presence of gravitation. Quantum field theory in Euclidean space-time discards the zero-point vacuum energy through the use of a normal time ordering procedure [5,20]. Because vacuum energy is gravitating and couples to all other field energies present at the quantum level, cancellation of the zero-point term is no longer possible when gravitational effects are significant. Likewise, this strong coupling regime of the far ultraviolet region suggests that even asymptotically free theories such as QCD reverse their properties in response to arbitrarily large non-perturbative effects. In fact, complex dynamics of quark-gluon plasma is expected to arise near the so-called transition temperature [16].

The non-linearity of the underlying field theory combined with the far-from-equilibrium dynamics induced by highly unstable vacuum fluctuations are prone to lead to self-organized criticality [6]. Because dynamical instabilities can develop on long time-scales, the macroscopic description of phenomena in terms of conventional differential operators breaks down. This is, in essence, the main argument for using fractal operators in the far TeV region of field theory and for the passage from ordinary to fractional dynamics [14,18,19,21]. Since application of fractals in contemporary physics has become far ranging, the interest in fractional dynamics has grown at a steady pace in the last decade. There is now a broad range of applications of fractional dynamics in research areas where fractal attributes of underlying processes and the onset of long-range correlations demand the use of fractional calculus. These areas include, but are not limited to, wave propagation in complex and porous media, models of systems with chaotic and pseudo-chaotic dynamics, random walks with memory, colored noise and pattern formation, anomalous transport and Levy flights, studies of scaling phenomena and critical behavior, plasma physics, turbulence, quantum field theory, far-from-equilibrium statistical models, complex dynamics of data networks and so on (for a brief review of current applications, see [21–26]).

3. Conventions and assumptions

(a) Einstein summation convention is applied throughout.
(b) The Poincare index is denoted by $\mu = 0, 1, 2, 3$.
(c) We study a basic "toy" model containing a single pair of massive complex-scalar fields $\varphi(x)$, $\varphi^{*}(x)$.
(d) The analysis is carried out exclusively at the classical level. Suppression of quantum attributes and transition to classical behavior is the result of decoherence induced by steady exposure to large random fluctuations [27,28].

(e) Following [22–26] we use in our work the left Caputo fractional derivative defined as

$$D^{\alpha}\varphi(x) \doteq \frac{1}{\Gamma(n-\alpha)}\int_0^x \frac{\varphi^{(n)}(s)}{(x-s)^{\alpha+1-n}}\mathrm{d}s \tag{1}$$

where $n-1<\alpha<n$ and $\varphi^{(n)}(s) \doteq \mathrm{d}^n\varphi(s)/\mathrm{d}s^n$. We note that, in addition to using (1), many studies based on fractional calculus often start from alternative operators such as Riemann–Liouville and Grunwald–Letnikov derivatives and integrals (see [30–33] for details).

(f) Space–time variables and fields are suitably normalized as dimensionless observables.

(g) Assuming that the field dynamics has low-level fractionality, we use the so-called ε-expansion to perform the transition from first order to Caputo derivatives of order $\alpha \doteq 1-\varepsilon$ according to the prescription [22]

$$\begin{aligned} &D^{1-\varepsilon}\varphi(x) = \partial\varphi(x) + \varepsilon D_1\varphi(x) \\ &D_1\varphi(x) \doteq \partial\varphi(0)\ln|x| + \gamma\partial\varphi(x) + \int_0^x \partial^2\varphi(s)\ln|x-s|\mathrm{d}s \end{aligned} \tag{2}$$

4. Local scale invariance at the onset of fractional dynamics

The novel symmetry principle that underlies the onset of fractional dynamics in the TeV region is the *local scale invariance* of the theory [17]. There is a two-fold rationale for the onset of this symmetry, namely,

(a) Field dynamics is *scale-invariant*. This is equivalent to stating that, in dimensional regularization scheme, the outcome of the regularization procedure does not depend on the particular choice of $\varepsilon = 4-d$ [20].

(b) By analogy with the definition of the Lipschitz–Hölder exponent and to ensure compliance with relativity [14,17], we take the continuous dimension parameter ε to denote a *locally* defined function of space-time coordinates, $\varepsilon(x)$. In addition, we assume that $\varepsilon(x)$ may be expressed either as a *contravariant* $\varepsilon^i(x)$ or a *covariant* $\varepsilon_i(x)$ four-vector. This motivates us to formally extend (2) to

$$\begin{aligned} &D^{1-\varepsilon_\mu(x)}\varphi(x) = \partial^\mu\varphi(x) + \varepsilon^\mu(x)D_1^\mu\varphi(x) \\ &D_1^\mu\varphi(x) \doteq \partial^\mu\varphi(0)\ln|x| + \gamma\partial^\mu\varphi(x) + \int_0^x (\partial^2)^\mu\varphi(s)\ln|x-s|\mathrm{d}s \\ &D_{1-\varepsilon_\mu(x)}\varphi(x) = \partial_\mu\varphi(x) + \varepsilon_\mu(x)D_{1,\mu}\varphi(x) \\ &D_{1,\mu}\varphi(x) \doteq \partial_\mu\varphi(0)\ln|x| + \gamma\partial_\mu\varphi(x) + \int_0^x (\partial^2)_\mu\varphi(s)\ln|x-s|\mathrm{d}s \end{aligned} \tag{3}$$

5. Fractional dynamics of the complex-scalar field

The goal of this section is to show that the principle of local scale invariance and the introduction of fractional dynamics lead to a mechanism of gauge boson-fermion unification that is fundamentally distinct from the mechanism advocated by supersymmetry.

The Lagrangian density of our model is

$$L \doteq \partial_\mu\varphi(x)\partial^\mu\varphi^*(x) - m^2\varphi^*(x)\varphi(x) \tag{4}$$

where m the mass of the field and x is a shorthand notation for (x^μ) or (x_μ). Using the framework of Caputo derivatives and ε-expansion [22], one obtains

$$L \doteq D_\mu^{1-\varepsilon(x)}\varphi(x)D^{\mu,1-\varepsilon(x)}\varphi^*(x) - m^2\varphi^*(x)\varphi(x) \tag{5}$$

where the locally defined infinitesimal dimension $\varepsilon(x)$ satisfies the condition $\varepsilon(x)\cdot x \ll 1$. Our aim is to show that, in contrast with conventional field theory embodied in (4), use of Caputo derivatives guarantees invariance under local gauge transformations *without* the explicit need for gauge fields and covariant operators. To this end, let us perform the local phase change

$$\begin{aligned}&\varphi(x) \to \mathrm{e}^{\mathrm{i}\alpha(x)}\varphi(x)\\&\varphi^*(x) \to \mathrm{e}^{-\mathrm{i}\alpha(x)}\varphi^*(x)\end{aligned} \tag{6}$$

Up to a first-order approximation, Caputo derivatives transform as [22]

$$\begin{aligned}&D_\mu^{1-\varepsilon(x)}\varphi(x) \to D_\mu^{1-\varepsilon(x)}[\mathrm{e}^{\mathrm{i}\alpha(x)}\varphi(x)] = \partial_\mu[\mathrm{e}^{\mathrm{i}\alpha(x)}\varphi(x)] + \varepsilon(x)D^1_{1,\mu}[\mathrm{e}^{\mathrm{i}\alpha(x)}\varphi(x)]\\&D^{1-\varepsilon(x),\mu}\varphi^*(x) \to D^{1-\varepsilon(x),\mu}[\mathrm{e}^{-\mathrm{i}\alpha(x)}\varphi^*(x)] = \partial^\mu[\mathrm{e}^{-\mathrm{i}\alpha(x)}\varphi^*(x)] + \varepsilon(x)D_1^{1,\mu}[\mathrm{e}^{-\mathrm{i}\alpha(x)}\varphi^*(x)]\end{aligned} \tag{7}$$

where

$$\begin{aligned}&D^1_{1,\mu}(x) \doteq \ln|x|\partial_\mu[\mathrm{e}^{\mathrm{i}\alpha(0)}\varphi(0)] + \int_0^x \partial_\mu^2[\mathrm{e}^{\mathrm{i}\alpha(\lambda)}\varphi(\lambda)]\ln|x-\lambda|\mathrm{d}\lambda + \gamma\partial_\mu[\mathrm{e}^{\mathrm{i}\alpha(x)}\varphi(x)]\\&D_1^{1,\mu}(x) \doteq \ln|x|\partial^\mu[\mathrm{e}^{-\mathrm{i}\alpha(0)}\varphi^*(0)] + \int_0^x \partial^{2,\mu}[\mathrm{e}^{-\mathrm{i}\alpha(\lambda)}\varphi^*(\lambda)]\ln|x-\lambda|\mathrm{d}\lambda + \gamma\partial^\mu[\mathrm{e}^{-\mathrm{i}\alpha(x)}\varphi^*(x)]\end{aligned} \tag{8}$$

in which γ stands for the Euler constant and

$$\partial_\mu[\mathrm{e}^{\mathrm{i}\alpha(x)}\varphi(x)] = \mathrm{e}^{\mathrm{i}\alpha(x)}[\partial_\mu + \mathrm{i}\partial_\mu\alpha(x)]\varphi(x) \tag{9}$$

Local gauge invariance of (5) is preserved if Caputo derivatives transform covariantly, that is

$$\begin{aligned}&D_\mu^{1-\varepsilon(x)}[\mathrm{e}^{\mathrm{i}\alpha(x)}\varphi(x)] = \mathrm{e}^{\mathrm{i}\alpha(x)}D_\mu^{1-\varepsilon(x)}\varphi(x)\\&D^{\mu,1-\varepsilon(x)}[\mathrm{e}^{-\mathrm{i}\alpha(x)}\varphi^*(x)] = \mathrm{e}^{-\mathrm{i}\alpha(x)}D^{\mu,1-\varepsilon(x)}\varphi^*(x)\end{aligned} \tag{10}$$

On account of (7) and (10), we arrive at the following set of conditions:

$$\begin{aligned}&\mathrm{i}\mathrm{e}^{\mathrm{i}\alpha(x)}\varphi(x)\partial_\mu\alpha(x) = \varepsilon(x)\{D^1_{1\mu}[\mathrm{e}^{\mathrm{i}\alpha(x)}\varphi(x)] - \mathrm{e}^{\mathrm{i}\alpha(x)}D^1_{1\mu}\varphi(x)\}\\&(-\mathrm{i})\mathrm{e}^{-\mathrm{i}\alpha(x)}\varphi^*(x)\partial_\mu\alpha(x) = \varepsilon(x)\{D_1^{1,\mu}[\mathrm{e}^{-\mathrm{i}\alpha(x)}\varphi^*(x)] - \mathrm{e}^{-\mathrm{i}\alpha(x)}D_1^{1,\mu}\varphi^*(x)\}\end{aligned} \tag{11}$$

The direct consequence of (11) is that gauge fields *are no longer required* in a field theory built on fractional dynamics. The compensating role of the vector bosons is played by the *continuous dimension parameter* $\varepsilon(x)$. This conclusion is consistent with previous studies [14,17,19] and points to a novel unification mechanism of gauge boson and fermion fields, including classical gravitation. This mechanism is fundamentally different from the unification scheme postulated by supersymmetry and related quantum field models [20].

6. Emergence of massive field theories

It is known that, allowing elementary particles to have non-zero masses in quantum field theory violates local gauge and weak isospin symmetries imposed on the standard model lagrangian. The mechanism of so-called spontaneous symmetry breaking (SSB) posits that the vacuum itself acquires a non-zero charge distribution that leaves the Lagrangian invariant and generates both fermion and vector boson masses [5,20]. In SM, massive fermions exist in both left-handed and right-handed states. The only Dirac field operators that yield a non-vanishing mass are bilinear products of fields having the form

$$m\bar{\psi}\psi = m(\bar{\psi}_\mathrm{R}\psi_\mathrm{L} + \bar{\psi}_\mathrm{L}\psi_\mathrm{R}) \tag{12}$$

However, such mass terms mix right and left-handed spinors and are forbidden from the Lagrangian on account of violation of the weak isospin symmetry [5,20]. Stated differently, since ψ_L represents a SU(2) doublet and ψ_R a SU(2) singlet, the product of the two is not a singlet, as it ought to be in order to preserve the weak isospin symmetry. An immediate question that arises from the previous section is whether or not SSB still exists in a field theory based on fractional dynamics. Stated differently, can mass terms be introduced in the Lagrangian without violating local and weak isospin symmetries? To answer this question, we note that the first-order Caputo operator may be defined either from the ‘‘left’’ or from the ‘‘right’’ and, in general, the effect produced by $D^{1-\varepsilon(x)}$ is not identical with the effect produced by $D^{1+\varepsilon(x)}$. It follows that the proper description of fractional differentiation requires a *doublet* of Caputo operators $\begin{pmatrix} D^{1-\varepsilon_L(x)} \\ D^{1+\varepsilon_R(x)} \end{pmatrix}$ and a *doublet* of scalars $\begin{pmatrix} \varepsilon_L(x) \\ \varepsilon_R(x) \end{pmatrix}$ with $\varepsilon_{L,R}(x) \cdot x \ll 1$. Therefore, mass terms that correspond to Dirac bilinears assume the form:

$$\begin{aligned} &\psi \to D^{-\varepsilon_L}\psi \quad \text{or} \quad \psi \to D^{\varepsilon_R}\psi \quad \text{for singlets} \\ &(\psi_1 \quad \psi_2) \to \begin{pmatrix} D^{-\varepsilon_L} \\ D^{+\varepsilon_R} \end{pmatrix} (\psi_1 \quad \psi_2) \quad \text{and} \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to (D^{-\varepsilon_L} \quad D^{+\varepsilon_R}) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad \text{for doublets} \end{aligned} \tag{13}$$

It can be seen that these mass terms automatically preserve the weak isospin symmetry in a similar manner in which the Higgs scalar doublet works in the electroweak model [5,20,29].

7. Critical behavior in continuous dimension

The previous sections have shown that the concept of dimension takes on a key role in the far ultraviolet region of field theory. Here we elaborate on this conjecture by making connection to the philosophy of the renormalization group and critical behavior in continuous dimension [1,2]. To streamline the derivation, we refer in what follows to the original Lagrangian (4). Let us start by adding a potential term to (4), that is

$$L \doteq \partial_\mu \varphi(x) \partial^\mu \varphi^*(x) - m^2 \varphi^*(x)\varphi(x) + \lambda^2 [\varphi^*(x)\varphi(x)]^2 \tag{14}$$

Here $g = \lambda^2$ represents the self-interaction strength of the field. According to the renormalization group, the so-called beta-function defines how g "flows" with the sliding energy scale μ, that is

$$\beta(g) \doteq \frac{\mathrm{d}(g)}{\mathrm{d}t} \tag{15}$$

where

$$t \doteq \ln\left(\frac{\mu}{\mu_0}\right) \tag{16}$$

for an arbitrary reference scale μ_0. In the basin of attraction of a critical point t_c the field correlation length scales as

$$\xi \approx |t - t_c|^{\nu} \tag{17}$$

Here, critical exponent ν is given by [1,2]

$$\nu^{-1} = -\left.\frac{\partial \beta}{\partial(g)}\right|_{g=g^*} \tag{18}$$

and g^* stands for a fixed point of the beta-function, $\beta(g^*) = 0$.

One can exploit the interchangeable roles played by the sliding scale t and the dimension parameter $d(x)$ as follows. Assume that d_c represents the critical dimension for which g flows into the fixed point g^*. The mass of the complex-scalar field is known to be inversely proportional to the divergent correlation length and vanishes identically at the fixed point [1,2]

$$m[g^*(d_c), d_c] = 0 \tag{19}$$

In the basin of attraction of g^* the field develops mass according to the power law

$$m[g^*(d_c), d(x) - d_c] \approx |d(x) - d_c|^{\nu(d_c)} = |\varepsilon(x)|^{\nu(d_c)} \tag{20}$$

where

$$\nu^{-1}(d_c) = -\left.\frac{\partial \beta}{\partial(g)}\right|_{g^*(d_c)} \tag{21}$$

We are led to conclude that, as the fixed point is asymptotically approached and the continuous space-time dimensionality collapses to $d(x) \to d_c = 1, 2, 3, 4$, the complex-scalar field becomes massless, in agreement with conventional quantum field theory. Numerical analysis yields $\nu(d_c) = 0.5$ for $d_c = 1, 2, 3, 4$ which is found to match well the value reported in the literature [1,2].

An important observation is now in order. Following the universal properties of the RG flow near the onset of chaos in low-dimensional maps, the dimensional control parameter $\varepsilon(x) = |d_c - d(x)|$ is expected to asymptotically approach the critical value $\varepsilon_\infty = 0$ according to the geometric progression [15]:

$$\varepsilon_n(x) - \varepsilon_\infty \approx a_n(x) \cdot \delta^{-n} \tag{22}$$

in which $n \gg 1$ is the index defining the number of iteration steps, δ stands for a scaling constant that is representative for the class of dynamical maps under consideration and $a_n(x)$ is a coefficient which becomes asymptotically independent of n and x, that is, $a_\infty = a$. Substituting (22) in (20) produces to the mass scaling series

$$m_n[g^*(d_c), d_c - d_n(x)] \approx [a_n \delta^{-n}]^{1/2} \tag{23}$$

We may go a step further and state that, given the generic link between the coupling flow and the corresponding flows of masses and fields in RG [5,20,29], similar scaling pattern develops for g_n and the underlying fields of the theory, η_n. We thus expect to obtain, for $n \gg 1$

$$\begin{aligned} g_n - g^*(d_c) &\approx \delta^{-\lambda(d_c)n} \\ \eta_n - \eta^*(d_c) &\approx \delta^{-\zeta(d_c)n} \end{aligned} \tag{24}$$

where $\lambda(d_c)$ and $\zeta(d_c)$ represent two additional critical exponents dependent on d_c.

8. Universal scaling of fermion masses

Period-doubling bifurcations are defined by $n = 2^m$, with $m > 1$ [15]. Replacing in (23) yields the following mass series:

$$m_m(x) \approx \sqrt{a_{2^m}(x)} \cdot \delta^{-2^{m-1}} \tag{25}$$

where $\delta = 4.669\ldots$ represents the Feigenbaum constant for the onset of chaos in quadratic maps. The ratio of two arbitrary masses is therefore

$$\frac{m_l(x)}{m_m(x)} \approx \sqrt{\frac{a_{2^l}(x)}{a_{2^m}(x)}} \cdot \frac{\delta^{-2^{l-1}}}{\delta^{-2^{m-1}}} \tag{26}$$

where $\lim \frac{a_l}{a_m} = 1$ as $l, m \to \infty$. Thus, for two sufficiently distant consecutive terms in the mass series, the dependence of $a_n(x)$ on the space-time variable may be suppressed and we obtain

$$\frac{m_l}{m_{l+1}} \approx \sqrt{\frac{a_{2^l}}{a_{2^{l+1}}}} \cdot \delta^{2^{l-1}} = \sqrt{a_{2^l, 2^{l+1}}} \cdot \delta^{2^{l-1}} \tag{27}$$

It is important to emphasize that (27) provides only a *first-order approximation* considering that (a) (27) is less accurate if the iteration index is not large enough, that is, if $l \approx O(1)$, (b) there is a fair amount of uncertainty involved in determining the quark mass spectrum [11]. Numerical results derived from (27) are displayed in Table 1. This table contains a side-by-side comparison of estimated versus actual mass ratios for charged leptons and quarks and a similar comparison of gauge coupling ratios. All masses and couplings are evaluated at the energy scale given by the top quark mass. Quark masses are averaged using the most recent reports issued by the Particle Data Group [11]. Specifically, $m_u = 2.12$ MeV; $m_d = 4.22$ MeV; $m_s = 80.9$ MeV; $m_c = 630$ MeV; $m_b = 2847$ MeV; $m_t = 170{,}800$ MeV.

The scaling sequence of charged leptons and quarks may be graphically summarized with the help of the following diagrams:

Table 1
Fermion masses and coupling ratios

Scaling ratio	2^{l-1}	Experimental value	Estimated value
m_u/m_c	4	3.365×10^{-3}	2.104×10^{-3}
m_c/m_t	4	3.689×10^{-3}	2.104×10^{-3}
m_d/m_s	2	0.052	0.046
m_s/m_b	2	0.028	0.046
m_e/m_μ	4	4.745×10^{-3}	2.104×10^{-3}
m_μ/m_τ	2	0.061	0.046
$(\alpha_{EM}/\alpha_W)^2$	2	0.045	0.046
$(\alpha_{EM}/\alpha_s)^2$	4	2.368×10^{-3}	2.104×10^{-3}

$$\overbrace{d \quad s}^{\delta^2} \qquad \overbrace{s \quad b}^{\delta^2}$$

$$\overbrace{u \quad c}^{\delta^4} \qquad \overbrace{c \quad t}^{\delta^4}$$

$$\overbrace{e \quad \mu}^{\delta^4} \qquad \overbrace{\mu \quad \tau}^{\delta^2}$$

$$\overbrace{\alpha_{EM}^2 \quad \alpha_W^2}^{\delta^2} \qquad \overbrace{\alpha_W^2 \quad \alpha_s^2}^{\delta^2}$$

Based on the above scheme, one may infer that neutrinos masses are arranged according to the possible pattern:

$$\overbrace{\nu_e \quad \nu_\mu}^{\delta^{16}} \qquad \overbrace{\nu_\mu \quad \nu_\tau}^{\delta^4}$$

It is also instructive to note that quarks and charged leptons follow a different period doubling path. To this end, let us organize the charged lepton and quark masses in a collection of triplets, that is

$$m_l \doteq [\, m_e \quad m_\mu \quad m_\tau \,], \quad m_q \doteq \begin{bmatrix} m_u & m_d \\ m_c & m_s \\ m_t & m_b \end{bmatrix} \tag{28}$$

It can be seen that the mass scaling for adjacent quarks stays constant within either one of the triplets (u, c, t) or (d, s, b), whereas the mass scaling for charged leptons varies as a geometric series in δ^2 within the triplet (e, μ, τ). This finding points out toward a symmetry breaking mechanism that segregates lepton and quark phases in the process of cooling from the far ultraviolet region of field theory to the low-energy region of SM.

9. Concluding remarks

We have argued that fractional dynamics represents an analytic framework suitable for the description of physical phenomena that are likely to arise in the TeV realm of particle physics. Unlike conventional quantum field theory, fractional dynamics describes far-from-equilibrium statistical processes that give rise to manifest scale invariance, non-local correlations and extensive symmetry breaking. Using fractional dynamics and the benchmark example of complex-scalar field theory, we have explored the potential spectrum of phenomena that may to emerge beyond the energy range of SM. Based on this framework, we have shown that, near the asymptotic boundary of field theory, (a) gauge bosons and fermions become unified through a fundamentally different mechanism than the one advocated by supersymmetry; (b) SSB and the emergence of massive field theories occur as a result of critical behavior in continuous dimension; (c) particles develop a family structure that is tied to the universal transition to chaos in unimodal maps. First-order predictions were found to match reasonably well current experimental data. However, as pointed out in Section 1, our goal is not to formulate a comprehensive solution to the host of open challenges surrounding SM. Concurrent research efforts are needed to confirm or falsify these preliminary findings. In particular, the long-awaited operation of the Large Hadron Collider and similar high-energy accelerator sites should soon produce experimental evidence that backs or disproves our model.

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Fractional dynamics and the TeV regime of field theory

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Abstract

Fractional dynamics offers a reliable tool for the study of far-from equilibrium processes that display scale-invariant properties, dissipation and long-range correlations. This is particularly attractive when dealing with the complex dynamics generated in the deep ultraviolet regime of quantum field theory. We analyze a simple scalar field Lagrangian using Caputo derivatives and the approximation of low-level fractionality. Results may be extrapolated to more realistic field models and suggest a series of surprising implications regarding phenomena that are expected to emerge beyond the range of the standard model for particle physics.

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Keywords: Fractional dynamics; Unified field theory; Lagrangian field theory; Standard model

1. Introduction

Fractional derivatives are an effective tool for describing the non-trivial behavior of complex phenomena whose dynamics is far-from equilibrium and cannot be characterized using traditional *analytic* functions. Often, this behavior is caused by coupling the underlying physical system to a reservoir of steep and highly correlated fluctuations. Owing it to their manifest scale-invariant attributes, *fractals* and *multi-fractals* provide a suitable framework for the study of a large class of non-analytic functions and fields [1,2]. For example, fractal geometry can be successfully associated with random walk models (fractional Brownian motion) or with the onset of criticality in statistical physics and fluid dynamics (phase transitions in spin clusters and fully developed turbulence). Recent years have demonstrated that *fractional dynamics* provides a natural framework for mapping out the evolution in fractal environments and modeling the physics of systems having

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multiple scales [1,3–8]. It is for this reason that fractional dynamics is also beneficial when exploring the rich spectrum of complex processes that is likely to arise in the deep ultraviolet regime of quantum field theory. Taking advantage of this capability, we analyze in this work a simple scalar field Lagrangian using Caputo derivatives and the approximation of low-level fractionality. Results may be extrapolated to more realistic field models and suggest a series of surprising implications regarding phenomena that are expected to emerge beyond the range of the standard model for particle physics.

We emphasize that our work has an introductory nature and, as such, it is not aimed to be either entirely rigorous or formally complete. Additional research is needed to confirm, develop or disprove our preliminary findings.

The paper is organized in the following way: Sections 2 and 3 introduce the main assumptions and rationale for fractional dynamics in the deep ultraviolet domain of field theory. Section 4 highlights the link between fractional dynamics and general relativity. Variational formulation of fractional dynamics as applied to field theory is discussed in Section 5. A strategy for field unification in the TeV regime using fractional dimension is investigated in the next two sections. The dynamic connection between fractional dimension and spin is elaborated upon in Section 8. Last section details how fractional dynamics may be used to motivate breaking of parity and time-reversal symmetries. The paper concludes with a brief summary of main results and future challenges.

2. Notation and conventions

(a) Einstein summation convention is applied throughout. Poincare indices are denoted by $i,j,k = 0,1,2,3$ and $SU(2)$, $SU(3)$ group indices by a and b, respectively.

(b) the analysis is carried out exclusively at the classical level. Suppression of quantum attributes and transition to classical behavior is the result of decoherence induced by steady exposure to large random fluctuations [9,10]. A conceptual benefit of this ansatz is that field theory built on fractional dynamics is free from any quantum or chiral anomalies.

(c) we follow the rationale of [3] and use in our work the left Caputo fractional derivative defined as

$$D^{\alpha}\varphi(x) \doteq \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{\varphi^{(n)}(s)}{(x-s)^{\alpha+1-n}} \, \mathrm{d}s \tag{1}$$

where $n-1<\alpha<n$ and $\varphi^{(n)}(s) \doteq \mathrm{d}^n\varphi(s)/\mathrm{d}s^n$.

(d) to avoid confusing (1) with the covariant derivative, the latter operator is generically represented as

$$\mathscr{D}^i\varphi(x) \doteq \partial^i\varphi(x) + \text{gaugeterms} \tag{2}$$

(e) space–time variables and fields are suitably normalized as dimensionless observables.

3. Fractals and the deep ultraviolet region of field theory

It is generally believed that quantum field theory breaks down near the so-called Cohen–Kaplan threshold of ~100 TeV as a result of exposure to large vacuum fluctuations and strong-gravitational effects. No convenient redefinition of observables is capable of turning off the dynamic contribution of these effects. For instance, it is known that the zero-point vacuum energy diverges quadratically in the presence of gravitation. Quantum field theory in Minkowski space–time discards the zero-point vacuum energy through the use of a normal time ordering procedure [11]. Because vacuum energy is gravitating and couples to all other field energies present at the quantum level, cancellation of the zero-point term is no longer possible when gravitational effects are significant. Our previous considerations suggest, in this context, that fractal geometry and fractional dynamics assume a leading role in the description of TeV physics [12,13]. For reasons that will become clear later on, we briefly review below the concept of *fractional dimension* and its relationship to the *index of fractional differentiation*.

A prototype of a continuous but nowhere differentiable field is the generalized Weierstrass function (GWF)

$$W(x) \doteq \sum_{n=-\infty}^{\infty} \frac{(1-\mathrm{e}^{\mathrm{i}\gamma_n x})\mathrm{e}^{\mathrm{i}\phi_n}}{\gamma^{(2-D)n}} \tag{3}$$

where x denotes the space coordinate, $\gamma > 1$ and ϕ_n stands for a series of arbitrary phases [1]. The fractional dimension of GWF is $1 < D < 2$, or

$$[W(x)] = D \tag{4}$$

and is considered to be a measure of its *degree of irregularity*. It can be shown that the correlation function of GWF is represented by

$$C(s) \doteq \left\langle |W(x+s) - W(x)|^2 \right\rangle_\phi \sim s^{2(2-D)} \tag{5}$$

This expression indicates that, in addition to being a measure of irregularity, D determines the *range over which field correlations extend*. In the above, the average is taken over an ensemble of realizations of ϕ_n phases that is uniformly distributed over the interval $(0, 2\pi)$. Carrying out a fractional derivation on GWF with the help of the Caputo operator (1), has the effect of increasing the fractional dimension D to D' according to

$$D' = [D^\alpha W(x)] = [W(x)] + \alpha = D + \alpha \tag{6}$$

It is apparent from (6) that the index of fractional differentiation α is equivalent to a linear shift in fractional dimension, i.e. $\alpha = D' - D = \Delta D$.

Fractal geometry of the underlying space–time manifold may be introduced by analogy with these definitions. In particular, space–time may be regarded as a fractal four-vector whereby each component x^i, $i = 0, 1, 2, 3$ represents a GWF. For instance we write, in parametric form

$$x^i \doteq \sum_{n=-\infty}^{\infty} \frac{(1-\mathrm{e}^{\mathrm{i}\gamma_n \tau})\mathrm{e}^{\mathrm{i}\phi_n}}{\gamma^{(2-d^i)n}} \tag{7}$$

where d^i are fractional dimensions assigned to each coordinate. In this case, the index of fractional differentiation gets upgraded to the four-vector $\alpha^i = d'^i - d^i = \Delta d^i$.

4. Fractional dynamics and classical gravity

Consider a model describing the dynamics of a scalar field $\varphi(x) \doteq \varphi(x_i)$ embedded in 3 + 1 Euclidean space–time. The generic Lagrangian associated with this model is given by

$$L(\varphi, \partial\varphi) \doteq \eta^{ij}\partial_i\varphi\partial_j\varphi - U(\varphi) = (\partial\varphi)^2 - U(\varphi) \tag{8}$$

Here, the Minkowski metric η^{ij} has signature $(+, -, -, -)$ so that $\eta^{00} \doteq +1$, $U(\varphi)$ represents the potential energy density and

$$(\partial\varphi)^2 \doteq \partial_i\varphi\partial^i\varphi = (\partial\varphi/\partial x_0)^2 - \sum_k (\partial\varphi/\partial x_k)^2 \tag{9}$$

for $k = 1, 2, 3$. Assuming that the field dynamics has low-level fractionality, we may use the so-called ε-expansion to perform the transition from first-order to Caputo derivatives of order $\alpha \doteq 1 - \varepsilon$ according to the prescription [3]

$$\begin{aligned} D^{1-\varepsilon}\varphi(x) &= \partial\varphi(x) + \varepsilon D_1\varphi(x) \\ D_1\varphi(x) &\doteq \partial\varphi(0)\ln|x| + \gamma\partial\varphi(x) + \int_0^x \partial^2\varphi(s)\ln|x-s|\,\mathrm{d}s \end{aligned} \tag{10}$$

where γ stands for the Euler constant and $\varepsilon x \ll 1$. By analogy with the definition of the Lipschitz–Hölder exponent [14,15], we turn our attention to the most general case where the fractional index ε represents a *locally* defined function of space–time coordinates. In addition, we assume that the fractional index may be expressed either as a *contravariant* $\varepsilon^i(x)$ or a *covariant* $\varepsilon_i(x)$ four-vector. This motivates us to formally extend (10) to

$$
\begin{aligned}
&D^{1-\varepsilon_i(x)}\varphi(x) = \partial^i\varphi(x) + \varepsilon^i(x)D_1^i\varphi(x) \\
&D_1^i\varphi(x) \doteq \partial^i\varphi(0)\ln|x| + \gamma\partial^i\varphi(x) + \int_0^x (\partial^2)^i\varphi(s)\ln|x-s|\,\mathrm{d}s \\
&D_{1-\varepsilon_i(x)}\varphi(x) = \partial_i\varphi(x) + \varepsilon_i(x)D_{1,i}\varphi(x) \\
&D_{1,i}\varphi(x) \doteq \partial_i\varphi(0)\ln|x| + \gamma\partial_i\varphi(x) + \int_0^x (\partial^2)_i\varphi(s)\ln|x-s|\,\mathrm{d}s
\end{aligned} \tag{11}
$$

By analogy with the standard notation for integer-order operators, the first couple of fractional derivatives are contravariant whereas the second couple denotes covariant derivatives. On this basis, the most straightforward generalization of (8) on account of (11) gives

$$L(\varphi, D\varphi) \doteq D_{1-\varepsilon_i(x)}\varphi D^{1-\varepsilon_i(x)}\varphi - U(\varphi) \tag{12}$$

To further simplify calculations we set

$$\partial^i\varphi(0) = \partial_i\varphi(0) = \partial^{2,i}\varphi(x) = \partial_{2,i}\varphi(x) = 0 \tag{13}$$

Replacing (11) in (12) we obtain

$$L(\varphi, \partial\varphi) = g_{i,\varepsilon}^i(x)\partial_i\varphi\partial^i\varphi - U(\varphi) + O(\varepsilon^2) \tag{14}$$

in which the metric coefficients are defined as

$$\boxed{g_{i,\varepsilon}^i(x) \doteq 1 + \gamma[\varepsilon_i(x) + \varepsilon^i(x)]} \tag{15}$$

We are led to conclude that the effect of low-order fractional derivatives is to transform the Minkowski metric into a Riemann metric determined by $g_{i,\varepsilon}^i(x)$. Stated differently, enabling the original Lagrangian to contain fractional derivatives is physically equivalent to embedding the field in non-Euclidean space–time. This finding is consistent with ideas developed in [12,13].

An important observation is now in order. As (5) indicates, the fractional index $\varepsilon^i(x)$ defines the range of field correlations in space and time. Fractional dynamics is essentially a non-local theory whereby phenomena evolve on multiple scales that are coupled to each other [1,12]. It is precisely this *scale coupling* that makes the metric field $g_{i,\varepsilon}^i(x)$ a self-interacting entity and justifies its deep connection to general relativity and the physics of classical gravitation.

Regarding (15), we note that cartesian frames of references are characterized by $\varepsilon^i(x) = \varepsilon_i(x)$. We also note that (15) reduces to the standard Minkowki metric in two distinct cases, i.e. (a) vanishing fractional index, $\varepsilon^i(x) = \varepsilon_i(x) = 0$, (b) anti-symmetric components of the fractional index, that is $\varepsilon^i(x) = -\varepsilon_i(x)$.

5. Fractional dynamics and the action principle

To further carry out our analysis it is convenient to introduce the following hypotheses:

(a) fractional index derives from a locally defined potential function, that is

$$\varepsilon^i(x) = \partial^i\xi(x), \qquad \varepsilon_i(x) = \partial_i\xi(x) \tag{16}$$

(b) (14) is considered a Lagrangian depending on two independent scalar fields embedded in $3+1$ space–time, namely $L \doteq L(\partial^i\xi, \varphi, \partial\varphi)$. The action functional is then

$$S[\xi, \varphi] \doteq \int_M L(\partial^i\xi, \varphi, \partial\varphi)\mathrm{d}x \tag{17}$$

from which the field equations follow as a result of Hamilton's principle.

6. Emergence of field charges and gauge-free theories

An intrinsic variation of field variables that does not involve any coordinate transformation is given by

$$\delta x^i = 0, \quad \delta\varphi = \overline{\varphi}(x) - \varphi(x), \quad \delta\xi^i(x) = \overline{\xi^i}(x) - \xi^i(x) \tag{18}$$

The conserved four-vector current corresponding to this variation takes the form [16,17]

$$j^i = \frac{\partial L}{\partial \varepsilon^i}\delta\xi^i(x) + \frac{\partial L}{\partial(\partial^i\varphi)}\delta\varphi = 2\gamma[\partial_i\varphi\partial^i\varphi\delta\xi^i + \varepsilon^i\partial_i\varphi\delta\varphi] \tag{19}$$

such that $\partial^i j^i = 0$. In very broad terms, this relationship asserts that $\varepsilon^i(x)$ gives rise to a supplementary contribution to the conserved four-vector current generated from internal field symmetries. As a result and as the next section shows, the existence of internal charges of field theory (such as electric charge, color, weak isospin as well as compensating gauge fields) may be interpreted as a direct *manifestation of fractional dynamics.*

In particular, let us consider a local gauge transformation that is applied to both fields $\varphi(x)$ and $\xi^i(x)$, that is

$$\delta\varphi(x) \doteq \frac{\partial\Lambda(x)}{\partial x}, \qquad \delta\xi^i(x) \doteq \frac{\partial\Omega^i(x)}{\partial x} \tag{20}$$

where $\Lambda(x)$ and $\Omega^i(x)$ are arbitrary real functions. We thus obtain

$$j^i = 2\gamma\left[\partial_i\varphi\partial^i\varphi\frac{\partial\Omega^i(x)}{\partial x} + \varepsilon^i\partial_i\varphi\frac{\partial\Lambda(x)}{\partial x}\right] \tag{21}$$

indicating that, in order to secure a vanishing divergence for the current (19), the contribution of the first gauge term in (21) is automatically balanced out by the contribution of the second term. It is seen that, in contrast with standard formulation of relativistic quantum field theory, invariance under local gauge transformations in fractional dynamics may be accomplished *without adding compensating gauge fields to the Lagrangian*. This key observation is in line with the framework discussed in [13,18], and suggests a natural mechanism for field unification beyond the standard model of particle physics. In the next section we introduce the basis for this unification program.

7. Asymptotic unification of classical gravity with gauge interactions

Refer again to (10) and (11). To streamline the derivation and capture the key point of the argument, we relax condition (13) and re-write (11) as

$$\begin{aligned} &D^{1-\varepsilon_i(x)}\varphi(x) = \partial^i\varphi(x) + \varepsilon^i(x)D_1^i\varphi(x) \\ &D_1^i\varphi(x) \approx \int_0^x (\partial^2)^i\varphi(s)\ln|x-s|\,\mathrm{d}s \end{aligned} \tag{22}$$

under the explicit assumption that the field $\varphi(x)$ is a rapidly varying function and

$$\partial^i\varphi(0)\ln|x| + \gamma\partial^i\varphi(x) \ll \int_0^x (\partial^2)^i\varphi(s)\ln|x-s|\,\mathrm{d}s \tag{23}$$

Higher order derivatives based on the ε-expansion may be formulated in analogous way. For instance,

$$D^{2-\varepsilon_i(x)}\varphi(x) = (\partial^2)^i\varphi(x) + \varepsilon^i(x)D_2^i\varphi(x) \tag{24}$$

where

$$\begin{aligned} &D_2^i\varphi(x) \approx \int_0^x (\partial^3)^i\varphi(s)\ln|x-s|\,\mathrm{d}s \\ &(\partial^2)^i\varphi(0)\ln|x| + \gamma(\partial^2)^i\varphi(x) \ll \int_0^x (\partial^3)^i\varphi(s)\ln|x-s|\,\mathrm{d}s \end{aligned} \tag{25}$$

Furthermore, for ultra-short space–time intervals $x \ll 1$ it is convenient to use the following approximations

$$\begin{aligned} D_1^i\varphi(x) &\approx x\ln\frac{|x|}{2}(\partial^2)^i\varphi\left(\frac{x}{2}\right) \approx Q(x)(\partial^2)^i\varphi(x) \\ D_2^i\varphi(x) &\approx Q(x)(\partial^3)^i\varphi(x) \end{aligned} \tag{26}$$

in which

$$Q(x) \doteq \frac{x\ln|x|}{4} \tag{27}$$

Replacing (26) in (22) we obtain, in operator language,

$$D^{1-\varepsilon_i(x)} = \partial^i - \varepsilon^i(x)Q(x)D^{2-\varepsilon_i(x)} + [\varepsilon^i(x)Q(x)]^2(\partial^3)^i \tag{28}$$

Note that, due to the presence of the logarithm in the expression of $Q(x)$, the product $\varepsilon^i(x)Q(x)$ cannot be considered an infinitesimal quantity, that is, $\varepsilon^i(x)Q(x) \neq O(\varepsilon^2)$.

It is straightforward to see that (28) may be recursively generalized on account of the relationship linking $D^{3-\varepsilon_i(x)}$ to $(\partial^3)^i$. We obtain

$$D^{1-\varepsilon_i(x)} = \partial^i - \varepsilon^i(x)Q(x)D^{2-\varepsilon_i(x)} + [\varepsilon^i(x)Q(x)]^2D^{3-\varepsilon_i(x)} - [\varepsilon^i(x)Q(x)]^3D^{4-\varepsilon_i(x)} + \cdots \tag{29}$$

The previous section has shown that the effect of fractional operators is to generate a gauge-free theory by discarding fields whose function is to maintain local gauge invariance. On this basis, one can assert that there is no physical distinction between (29) and the covariant derivative operators of relativistic quantum field theory, briefly discussed in Appendix B. We may therefore perform a term by term identification of (29) and (B.1) to arrive at the following set of operational relations

$$\begin{gathered} \boxed{-ig_1\frac{Y}{2}B^i \Leftrightarrow -\varepsilon^i(x)Q(x)D^{2-\varepsilon_i(x)}} \\ \boxed{-ig_2\frac{\tau_a}{2}W_a^i \Leftrightarrow [\varepsilon^i(x)Q(x)]^2D^{3-\varepsilon_i(x)}} \\ \boxed{-ig_3\frac{\lambda_b}{2}G_b^i \Leftrightarrow -[\varepsilon^i(x)Q(x)]^3D^{4-\varepsilon_i(x)}} \end{gathered} \tag{30}$$

Since $\varepsilon^i(x) \approx [g^i_{i,\varepsilon}(x) - 1]$ according to (15), the four-vector $\varepsilon^i(x)$ represents the deviation from the Minkowski metric of the equivalent gravitational field $g^i_{i,\varepsilon}(x)$ induced by fractional dynamics. It follows that (30) provides the motivation for the *asymptotic unification of classical gravity with the combined* $SU(3) \times SU(2) \times U(1)$ *gauge symmetry of the standard model for particle physics*. In different terms, (30) suggests that the three gauge groups of the standard model *break out from the topological concept of fractional dimension* d^i that was introduced in Section 1.

8. Fractional dynamics and the physical origin of spin

It is well known that spin represents a quantum mechanical observable that has no classical analogue. The object of this section is to show that the concept of spin and the concept of fractional dimension are closely related.

A good starting point is (14) from which field equations may derived on the assumption that $\varepsilon^i(x)$ is an independent four-vector field and $O(\varepsilon^2)$ terms may be ignored. We find

$$\begin{aligned} &\partial^i(\partial_i\varphi\partial^i\varphi) = 0 \\ &\partial^i\left(\frac{\partial L}{\partial(\partial^i\varphi)}\right) + \frac{\partial U}{\partial\varphi} = 0 \end{aligned} \tag{31}$$

It follows that the energy-momentum tensor is defined by

$$T^{ij} = \frac{\partial L}{\partial \varepsilon^i}\varepsilon^j + \frac{\partial L}{\partial(\partial^i \varphi)}\partial^j \varphi - L\delta_{ij} \tag{32}$$

or, taking into account (14),

$$T^{ij} = \widetilde{T}^{ij} + \gamma \partial^i \varphi \partial_i \varphi [\varepsilon^j - 2\varepsilon^i(\delta_{ij} + \eta^{ij})] \tag{33}$$

Comparing to (A7) and recalling the scalar nature of field φ, leads to the result that the second term of (33) may be thought of as being generated by an *intrinsic spin* variable. In different words, we may establish the following formal connection in reference to Appendix A

$$\gamma \partial^i \varphi \partial_i \varphi [\varepsilon^j - 2\varepsilon^i(\delta_{ij} + \eta^{ij})] \Longleftrightarrow \partial_k f^{ikj} \tag{34}$$

where f^{ikj} represents the spin contribution to the canonical energy-momentum tensor.

It is apparent from these considerations that fractional dynamics has the effect of turning a *scalar field* into a *spinor field*. From a phenomenological standpoint and on account of Noether's theorem, one can go further and infer that *spin is an observable that may be interpreted as the conserved charge associated with the local variation of fractal dimension* $\varepsilon^i(x)$. More generally, since $\varepsilon^i(x)$ is a continuous function of coordinates, it can be stated that (34) extends the ordinary half-integer or integer spins to a continuous spectrum of eigenvalues. A similar scenario is discussed in [12,13] in relation to the onset of complexity within the TeV regime of field theory.

We wish to elaborate on this last point in more detail. Consider again the conserved four-vector current (19) and impose an internal variation on $\varepsilon^i(x)$ that does not involve any change of coordinates or field $\varphi(x)$

$$\delta x^i = 0, \quad \delta\varphi = 0, \quad \delta\xi^i(x) \doteq \overline{\xi^i}(x) - \xi^i(x) \tag{35}$$

According to the general theory of internal symmetries in Lagrangian field models, $\delta\varepsilon^i(x)$ takes the form

$$\delta\xi^i(x) = \mathrm{i}\theta^{\varepsilon,i} X^i \xi^i(x) \tag{36}$$

where $\theta^{\varepsilon,i}$ stands for an infinitesimal rotation angle in parameter space and X^i denote the set of linearly independent generators associated with this rotation [17]. It can be shown that X^i satisfy Lie algebra, that is

$$[X^i, X^j] = \mathrm{i}c^{ijk} X^k \tag{37}$$

in which c^{ijk} represent the structure constants of the Lie group. The conserved Noether current generated from (19) is given by

$$j^i = \frac{\partial L}{\partial \varepsilon^i}\delta\xi^i(x) = \frac{\partial L}{\partial \varepsilon^i}\mathrm{i}\theta^{\varepsilon,i} X^i \xi^i(x) \tag{38}$$

Thus, up to some multiplicative constant factors, we obtain

$$j^i = 2\gamma \partial_i \varphi \partial^i \varphi \delta\xi^i(x) \cong X^i \xi^i(x) \partial_i \varphi \partial^i \varphi \tag{39}$$

subject to the continuity condition $\partial^i j^i = 0$. Taking into account that $\partial^i(\partial_i \varphi \partial^i \varphi) = 0$ yields

$$\partial^i j^i = 0 \Rightarrow \partial^i [X^i \xi^i(x)] = 0 \tag{40}$$

which implies that

$$\int_{\Omega_M} X^0 \xi^0(x) \mathrm{d}^3 x = \text{const.} \tag{41}$$

where Ω_M designates the spatial region of the integration domain. On the basis of previous discussion, we interpret the invariant charge (41) as temporal component of the four-dimensional spin current $X^i\xi^i(x)$.

To summarize this section, we conclude that invariance of the action functional under a continuous variation of space–time dimension leads to the emergence of a conserved charge, which we identify with *spin*. The crucial ingredient in this conjecture is that $\xi^i(x)$ is a *continuous function*, enabling *infinitesimal* transformations of space–time dimension. In contrast with the basic premise of supersymmetry and related dynamic models dealing with discrete spin transformations, this finding points to a fundamentally different route to boson–fermion unification [12,13,18].

9. Breaking of parity and time-reversal symmetries

Invariance of field equations to space and time reflections is well established in classical and quantum theory. In light of this fact, parity non-conservation and breaking of the time-reversal symmetry in weak interactions and kaon physics, respectively, are regarded as anomalies for which the standard model offers no explanation [19,20]. The purpose of this section is to suggest that fractional dynamics may provide important insights into the physical origin of these anomalies. Following Section 3, our underlying premise is that complexity becomes relevant in the ultra-short distance regime of both weak interaction and kaon decay channels. For the sake of concision and simplicity, here we restrict the discussion to parity and employ the notation $x \doteq x_k$, $k = 1, 2, 3$.

Parity acts as a unitary operator in classical field theory [11,19]. Its action is represented by

$$P\eta(x) \doteq \eta(-x) \tag{42}$$

Consider next (1) and set $n = 1$ and $\alpha = 1 - \varepsilon$, with $\varepsilon^i(x) = \varepsilon_i(x) = \varepsilon$. Hence

$$D^{1-\varepsilon}\eta(x) \doteq \frac{1}{\Gamma(\varepsilon)} \int_0^x \frac{\eta^{(1)}(x)}{(x-s)^{1-\varepsilon}} \mathrm{d}s \tag{43}$$

such that

$$P[D^{1-\varepsilon}\eta(x)] \doteq P[D^{1-\varepsilon}]P[\eta(x)] = P[D^{1-\varepsilon}]\eta(-x) \tag{44}$$

or

$$P[D^{1-\varepsilon}]\eta(-x) = \frac{[(-1)^{\varepsilon-1}]}{\Gamma(\varepsilon)} \int_0^{(-x)} \frac{\eta^{(1)}(-s)}{(x+s)^{1-\varepsilon}} \mathrm{d}s \tag{45}$$

where

$$(-1)^{\varepsilon-1} = \exp[\mathrm{i}\pi(\varepsilon - 1)] \tag{46}$$

On the other hand,

$$D^{1-\varepsilon}[P\eta(x)] = D^{1-\varepsilon}\eta(-x) = \frac{1}{\Gamma(\varepsilon)} \int_0^x \frac{\eta^{(1)}(-s)}{(x-s)^{1-\varepsilon}} \mathrm{d}s \tag{47}$$

It is seen from (45) and (46) that the effect of parity operator P on the Caputo derivative is to produce a fractional shift in phase. This phase-shift is proportional to the index ε and reduces to an integer when the dynamics becomes classical, that is, in the limit $\varepsilon \to 0$. As it is known, parity and first-order derivative operators anti-commute in the classical limit, that is,

$$\{P, \partial\}\eta(x) = P[\partial\eta(x)] + \partial[P\eta(x)] = 0 \tag{48}$$

In contrast, P no longer anti-commutes with the fractional derivative operator for $\varepsilon \neq 0$ and a distinction is made between the "left" and "right" directions. In different words,

$$\{P, D^{1-\varepsilon}\}\eta(x) \neq 0 \tag{49}$$

Following the same line of arguments and using (12), parity and Hamiltonian may be shown to become non-commuting operators in the fractional dynamics regime, i.e.

$$[P, H^{1-\varepsilon}]\eta(x) \doteq (PH^{1-\varepsilon} - H^{1-\varepsilon}P)\eta(x) \neq 0 \tag{50}$$

10. Summary and conclusions

The description of complex dynamics in the TeV regime of field theory warrants the transition from ordinary calculus on smooth manifolds to fractional differentiation and integration. We have suggested that this transition has important implications regarding phenomena that are anticipated beyond the energy range of the standard model for particle physics. In particular, we have argued that:

(a) fractional dynamics in Minkowski space–time is equivalent to field theory in curved space–time. This finding points out to a natural mapping of gravity onto the fractal topology of space–time in to the deep ultraviolet region of field theory.
(b) the three gauge groups of the standard model, as well as the spin observable, are rooted in the topological concept of fractional dimension.
(c) fractional dynamics is the underlying source of parity non-conservation in weak interactions and of the breaking of time-reversal invariance in processes involving neutral kaons.

Follow-up studies may be devoted to understanding the relationship between predictions and experimental data, the link between the fractional phase-shift (46) and anyon statistics [13,21] and the role of multi-fractal measures in the future development of field models.

Appendix A

For reader's convenience, we survey below some results of Lagrangian field theory that are relevant to our work. Following [16,17], consider a generic classical field described by a set of $m = 1, 2, \ldots, N$ real or complex functions $\eta^m(x)$. The field equations stemming from the action principle read

$$\frac{\partial L}{\partial \eta^m} - \frac{\partial}{\partial x^i}\left[\frac{\partial L}{\partial(\partial^i \eta^m)}\right] = 0 \tag{A.1}$$

where $L(\eta^m(x), \partial^i \eta^m(x), x^i)$ stands for the Lagrangian density. The infinitesimal Lorentz transformation of coordinates is given by

$$x' \doteq \widetilde{L}x \Rightarrow x'^i = x^i + \omega^i_j x^j + a^i \tag{A.2}$$

where a^i is a constant translation vector and $\omega^{ij} = -\omega^{ji}$ a constant anti-symmetric tensor describing rotations. In the most general case, the field components respond to the Lorentz transformation (A.2) as

$$\eta'^m(x') = [\delta^m_n + \Lambda^m_n(\widetilde{L})]\eta^n(x) \tag{A.3}$$

in which

$$\begin{aligned} &\Lambda^m_n = 0 \quad \text{if } \eta^m(x) \text{ is a scalar field} \\ &(1 + \Lambda^m_n) = \widetilde{L}^m_n \quad \text{if } \eta^m(x) \text{ is a vector field} \\ &(1 + \Lambda^m_n) = S(\widetilde{L})^m_n \quad \text{if } \eta^m(x) \text{ is a spinor field} \end{aligned} \tag{A.4}$$

Here, $\widetilde{L}^m_n$ acts as a Lorentz transformation operator and $S(\widetilde{L})^m_n$ the general spin representation of the Lorentz group. It can be shown that

$$\Lambda^m_n = \frac{1}{2} S^{mij}_n \omega_{ij} \tag{A.5}$$

in which S^{mij}_n denotes the spin tensor. Under these circumstances

$$\begin{aligned} &S^{mij}_n = 0 \quad \text{if } \eta^m(x) \text{ is a scalar field} \\ &S^{mij}_n = \eta^{mi}\eta^j_n - \eta^{mj}\eta^i_n \quad \text{if } \eta^m(x) \text{ is a vector field} \\ &S^{mij}_n = \frac{1}{4}(\gamma^i\gamma^j - \gamma^j\gamma^i)^m_n \quad \text{if } \eta^m(x) \text{ is a four-component spinor field} \end{aligned} \tag{A.6}$$

where γ^i is the set of Dirac matrices [11,22]. We obtain for the energy-momentum tensor of the field

$$T^{ij} = \widetilde{T}^{ij} - \frac{\partial}{\partial x^k} f^{ikj} = \left[\frac{\partial L}{\partial(\partial^i \eta^m)} \partial^j \eta^m - L\delta_{ij}\right] - \frac{\partial}{\partial x^k} f^{ikj} \tag{A.7}$$

Here, δ_{ij} is the Kronecker symbol and the spin term is defined as

$$f^{ikl} = \frac{1}{2} \frac{\partial L}{\partial(\partial^i \eta^m)} S^{mkl}_n \eta^n \tag{A.8}$$

In particular, the spin contribution vanishes if $\eta^m(x)$ represents a scalar field and the energy-momentum tensor assumes the canonical form

$$\widetilde{T}^{ij} = \frac{\partial L}{\partial(\partial^i \eta^m)} \partial^j \eta^m - L\eta^{ij} \tag{A.9}$$

Appendix B

A Lie group is a continuous group whose elements can be parameterized by a finite number of parameters. Of particular interest in relativistic quantum field theory are the Lie groups associated with the principle of gauge invariance, the so-called $SU(n)$ groups. Elements of the $SU(n)$ group are $n \times n$ unitary matrices with unit determinant. There are $n^2 - 1$ generators of the $SU(n)$ group represented by $n \times n$ traceless and Hermitian matrices whose elements satisfy the commutation relations of the Lie algebra.

The principle of gauge invariance demands that the action functional is left unchanged under global and local internal field transformations that generate three fundamental symmetries. These are the $SU(2) \times U(1)$ symmetry of electroweak interaction and the $SU(3)$ symmetry of strong interaction. The conserved global charges associated with these symmetries are the electric charge, the weak isospin and the QCD color, respectively. The non-abelian spin-1 fields that are introduced to secure local gauge invariance of the theory are the electromagnetic field, as well as the vector boson and gluon fields of the weak and strong interactions. As a result of demanding local gauge invariance, all ordinary derivatives entering the Lagrangian are upgraded to the so-called *covariant derivatives* including the set of gauge fields described above. The full covariant derivative operator of the standard model is written as [22]

$$\mathscr{D}^i \doteq \partial^i - \mathrm{i}g_1 \frac{Y}{2} B^i - \mathrm{i}g_2 \frac{\tau_a}{2} W_a^i - \mathrm{i}g_3 \frac{\lambda_b}{2} G_b^i \tag{B.1}$$

Here, g_1, g_2 and g_3 are the electroweak and strong coupling constants, Y is the generator of the $U(1)$ group, τ_a are Pauli matrices that generate the $SU(2)$ group and λ_b the generators of the $SU(3)$ group. Also, W_a^i and G_b^i stand for the weak and gluon fields, $a = 1, 2, 3$ is the $SU(2)$ group index and $b = 1, 2, 3, \ldots, 8$ the $SU(3)$ group index.

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Fractional dynamics, Cantorian space–time and the gauge hierarchy problem

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Abstract

The gauge hierarchy problem in particle physics refers to the large numerical disparity between the value of the Planck mass ($M_{Pl} \simeq 1.22 \times 10^{19}$ GeV) and the mass scale of the electroweak interaction ($M_{EW} \approx 10^2$ GeV). Explaining the hierarchy paradox has been attempted so far in quantum field models based on supersymmetry or higher dimensional space–time with a large number of extra dimensions (brane theories). Despite several years of experimental search, there is currently no validation for either one of these models. We approach the hierarchy paradox using the methodology of fractal operators in four-dimensional space–time. It is found that departure from the inverse-square gravity in the high-energy regime emerges naturally from the fractional Helmholtz equation and offers a simple resolution to the problem. Our work makes an explicit connection between the hierarchy problem and Cantorian geometry of space–time on energy scales comparable to the Planck mass.

1. Introduction

Relativistic quantum field theory is considered an effective representation of nature at low energies. It is generally accepted that any attempt to build a realistic field framework describing physics on large energy scales must account for the dynamic effect of the vacuum and for the strong gravitational effects induced by its fluctuations [27,23,29]. Gravitational stability of the vacuum sets a limit on the validity of any quantum field theory above a threshold energy scale [29,30]. Moreover, gravitationally bound states having masses of order $\sim 10^{19}$ GeV can be formed, in principle, on or below the Planck scale [5]. It has become increasingly clear in recent years that the adequate description of complex dynamics associated with generic random fluctuations demands use of fractal operators and fractional calculus [1–4]. From a mathematical physics viewpoint, introduction of this class of operators is closely related to a broad spectrum of topics ranging from analytic continuation [7] and non-local field theory [14] to generalized functions [15] and Levy stable probability distributions [16].

The gauge hierarchy problem in particle physics refers to the large numerical disparity between the value of the Planck mass ($M_{Pl} \simeq 1.22 \times 10^{19}$ GeV) and the mass scale of the electroweak interaction ($M_{EW} \approx 10^2$ GeV). Explaining the hierarchy paradox has been attempted in quantum field models based on supersymmetry or higher dimensional space–time with a large number of extra dimensions (brane theories). At the present time there is no empirical confirmation for either supersymmetry or brane theory. For example, experimental searches for departures from Newtonian gravity are focused on measuring the distance separating the branes, which is sought to be on the order of millimeters or less. No deviation from the inverse-square law has been found at distances as small as a tenth of a millimeter [19,20]. In light of these results, it appears that searching for alternate mechanisms that solve the gauge

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hierarchy problem is a worthwhile endeavor. We handle the hierarchy paradox using the methodology of fractal operators in four-dimensional space–time. It is found that departure from the inverse-square gravity in the high-energy regime emerges naturally from the fractional Helmholtz equation and offers a simple resolution to the problem. Results rely on the explicit connection between the hierarchy problem and Cantorian geometry of space–time on energy scales comparable to the Planck mass [17].

The plan of the paper is as follows: Section 2 introduces the gauge hierarchy problem. Models founded in higher dimensional space–time are briefly discussed in Section 3. Fractional solution of the Helmholtz equation is developed in Section 4. A link to fractional dynamics of the scalar field is established in Section 5. Scaling of the effective Planck mass and of the effective gravity coupling is the topic of Sections 6 and 7. Section 8 links results to the Cantorian space–time geometry. Concluding remarks are presented in Section 9.

2. The Planck and the gauge hierarchy problem

Let r be the distance at which the gravitational potential energy of a classical scalar particle of mass M equals rest its energy. Thus

$$G\frac{M^2}{r} = Mc^2 \tag{1}$$

where G stands for Newton's constant. If r is taken to represent the Compton wavelength of the particle, that is

$$r = \frac{\hbar}{Mc} \tag{2}$$

eliminating r and solving for M gives the Plank mass [6]

$$M_{\mathrm{Pl}} = (\hbar c G^{-1})^{1/2} \simeq 1.22 \times 10^{19}\ \mathrm{GeV} \tag{3}$$

At the same time, the electroweak interaction scale is set by either the vector boson mass ($M_{\mathrm{EW}} = M_W \approx 80$ GeV) or the Fermi constant ($M_{\mathrm{EW}} = G_{\mathrm{F}}^{-1/2} \approx 290$ GeV) [5]. The gauge hierarchy problem is defined by the large ratio of the Planck mass to the electroweak mass scale, which is on the order of 10^{17}.

3. Models with large or warped extra dimensions

Large extra spatio-temporal dimensions have been introduced in recent years as a possible solution to the hierarchy problem [9–13]. This scenario is based on extending space–time dimensionality to $D = 4 + n$, where gravity propagates in the extra dimensional bulk ($n \gg 1$) and the Standard Model fields are confined to the usual 3 + 1 space–time referred to as a "3-brane". According to this mechanism, the weakness of gravity in our four-dimensional space–time can be attributed to the spreading of gravity force lines into extra dimensions. This pervasive character of gravity enables coupling of the four dimensional continuum to neighboring branes. If the characteristic volume associated with the n-dimensional space is denoted by R^n, it can be shown that the effective four-dimensional Planck mass (M_{Pl}) is related to the Planck mass in D dimensions (M_S) via

$$M_{\mathrm{Pl}}^2 \sim R^n M_S^{n+2} \tag{4}$$

While the mechanism of large extra dimensions resolves the disparity between M_{Pl} and M_{EW}, it introduces a new hierarchy between the energy scale associated with extra-dimensional space ($\mu \sim 1/R$) and M_{EW} [11]. An alternative is provided by the model of warped extra dimensions [11,12]. In this conjecture, the four-dimensional metric is multiplied by an exponential "warp" factor that is a rapidly changing function of an additional extra dimension. Ramifications of this scenario are discussed in [12].

We now proceed with the development of our arguments. Based on the introductory remarks, a logical starting point is fractional generalization of the scalar equation describing field propagation in arbitrary dimensions [8]. Natural units are assumed throughout the remainder of the paper ($\hbar = c = 1$).

4. Fractional representation of the Helmholtz equation

Consider a classical scalar field $\varphi(\vec{r}, t)$ of mass m_0 where $\vec{r} = \{x_j\}$, $j = 1,2,3$. It satisfies the well-known wave equation [21,24]

$$\left[\left(\frac{\partial^2}{\partial t^2}-\nabla^2\right)+m_0^2\right]\varphi(\vec{r},t)=0 \tag{5}$$

Eq. (5) is otherwise known as the Klein–Gordon equation describing the space–time evolution of a free spinless particle of mass m_0. Performing the operator substitution

$$E\to \mathrm{i}\frac{\partial}{\partial t};\quad \vec{p}\to -\mathrm{i}\nabla \tag{6}$$

in (5) retrieves the relativistic dispersion relation

$$E^2=\vec{p}^2+m_0^2 \tag{7}$$

It is customary to use a dimensionless representation of (5) and (7). Dividing each side of these equations by an arbitrary mass scale M_0 and introducing the normalization prescription

$$\vec{r}^0=M_0\vec{r};\quad t^0=M_0t;\qquad \frac{\partial}{\partial t^0}=\frac{1}{M_0}\frac{\partial}{\partial t};\quad \frac{\partial}{\partial x_j^0}=\frac{1}{M_0}\frac{\partial}{\partial x_j};\quad \nabla_0=\left\{\frac{\partial}{\partial x_j^0}\right\} \tag{8}$$

leads to

$$\begin{aligned}&\left[\left(\frac{\partial^2}{\partial (t^0)^2}-\nabla_0^2\right)+(m_0^0)^2\right]\varphi(\overrightarrow{r^0},t)=0\\&(E^0)^2=(\vec{p}^0)^2+(m_0^0)^2\end{aligned} \tag{9}$$

in which

$$m_0^0=\frac{m_0}{M_0} \tag{10}$$

The Helmholtz equation is the time-independent version of (5) and (9). The Green function of the dimensionless Helmholtz equation satisfies [8,21]

$$\nabla_0^2G(\vec{r}^0;m_0^0)-(m_0^0)^2G(\vec{r}^0;m_0^0)=-\delta(\vec{r}^0) \tag{11}$$

where

$$\delta(\vec{r}^0)=\delta(x_1^0)\delta(x_2^0)\delta(x_3^0) \tag{12}$$

is the three-dimensional delta function representing a source-point located at the origin of the coordinate system. The explicit form of the Green function is

$$G(r^0;m_0^0)=\frac{\exp(-m_0^0r^0)}{4\pi r^0} \tag{13}$$

The high-energy regime implies that r^0 is taken well inside the effective range of the source ($r^0\ll 1/m_0^0$). In this case (13) decays as

$$\underset{m_0^0r^0\ll 1}{G(r^0;m_0^0)}\sim (r^0)^{-1} \tag{14}$$

It is well known that, since the Green function relates to the gravitational potential created by the delta source and the magnitude of a conservative force is the gradient of that potential, the above framework is the basis for the inverse-square law of classical field theory. We recall that both the delta and potential functions depend on the dimensionality of the embedding space; explicit forms for the one-dimensional, two-dimensional and higher-dimensional cases are known and well documented in the literature [8,25].

Motivated by the growing evidence for complexity in field theory, we now ask the following question: What happens if the geometry of the delta source is generalized from integer-dimensional to fractal? Stated differently, What is the form taken by the Green function when the delta source is effectively an intermediate between a three-dimensional and a four-dimensional distribution? The answer to this question is reported in [8] where it is shown that, if the embedding space is N-dimensional ($j=1,2,\dots,N$), the generalized form of the delta function is

$$\delta(x_1^0, x_2^0, \ldots, x_{N-1}^0, x_N^0) = \frac{\delta(x_1^0)\delta(x_2^0)\cdots\delta(x_{N-1}^0)(x_N^0)^{N-f-1}}{\Gamma(N-f)} \tag{15}$$

In the above expression f represents a real number between $N-1$ and N

$$N-1 < f < N \tag{16}$$

defining the intermediate space dimension. Well inside the effective range of the source $((r^0)_{N-1} \ll 1/m_0^0)$, it is found that the corresponding Green function decays as

$$\underset{m_0^0(r^0)_{N-1}\ll 1}{G_f(x_1^0, x_2^0 \ldots x_{N-1}^0; m_0^0)} \sim (r^0)_{N-1}^{\frac{1-f}{2}} \tag{17}$$

where

$$(r^0)_{N-1} = \sqrt{(x_1^0)^2 + (x_2^0)^2 + \cdots + (x_{N-1}^0)^2} \tag{18}$$

It follows that the fractional gravitational potential in free three-dimensional space is given by

$$\underset{m_0^0(r^0)_3\ll 1}{G_f(x_1^0, x_2^0, x_3^0; m_0^0)} \sim (r^0)_3^{\frac{1-f}{2}} \tag{19}$$

Let $M_{\text{eff,Pl}}$ denote the effective Planck mass representing the generalization of M_{Pl} in the context provided by fractional Green function (19). As before, let M_0 denote the reference mass scale. By analogy with (1) we have

$$(GM_0^2)\frac{\left(\frac{M}{M_0}\right)^2}{(r^0)_3^{\frac{1-f}{2}}} = \frac{M}{M_0} = M^0 \tag{20}$$

or, taking into account (3) and solving for the effective Planck mass $(M = M_{\text{eff,Pl}})$

$$M_{\text{eff,Pl}}^0 = (M_{\text{Pl}}^0)^2 (r^0)_3^{\frac{1-f}{2}} \tag{21}$$

in which

$$\begin{aligned} M_{\text{eff,Pl}}^0 &= \frac{M_{\text{eff,Pl}}}{M_0} \\ M_{\text{Pl}}^0 &= \frac{M_{\text{Pl}}}{M_0} \end{aligned} \tag{22}$$

In closing this section, we recall that the left-hand side of (1) corresponds to the inverse-square law of classical potential theory in three-dimensional space. It can be shown that the inverse-square law is rooted in the rotational invariance of the three-dimensional space at low energy scales [21]. In contrast, the non-trivial nature of space–time in the high-energy regime is expected to break the rotational symmetry of ordinary space and open the door for non-commutative geometry [22,23]. In general, breaking the rotational symmetry of ordinary space amounts to upgrading the inverse-square gravity to a power-law depending on a non-integer exponent.

5. Connection to fractional dynamics of the scalar field

As previously stated, physics on high-energy scales is characterized by large perturbations in momenta produced by vacuum fluctuations and their strong gravitational effects. The erratic spectrum of fluctuations generates steady deviations from the unitary time evolution of quantum mechanics and drives the transition from order to chaos. The onset of Hamiltonian chaos transforms the smooth topology of classical phase-space into an irregular and highly fragmented structure. Adequate modeling of this regime requires replacing the conventional differential operators of quantum mechanics with fractal operators, as discussed in [1,18,26]. In this context, an additional key observation is that, due to the environment-induced decoherence and suppression of interference, all particles and fields loose their quantum memory and become classical objects [31,32].

In [18], a fractional generalization of the Klein–Gordon field theory expressed by (9) has been presented. To make the paper self-contained and for reader's convenience, we reiterate the main arguments here.

It is known that the chaotic dynamics of Hamiltonian systems is conveniently described as a fractional diffusion process [1,26]. The space–time flow of the scalar field $\varphi(\vec{r}, i)$ is encoded in a fractional generalization of the conventional Klein–Gordon equation. The new equation depends on two non-integer exponents (α, β) describing the space and time differentiation of the field, respectively. According to this prescription, the ordinary space and time differentiation operators are extended to

$$\begin{aligned} &\frac{\partial}{\partial t} \to \frac{\partial^\beta}{\partial t^\beta} \\ &\frac{\partial}{\partial |x_j|} \to \frac{\partial^\alpha}{\partial |x_j|^\alpha} \end{aligned} \tag{23}$$

in which $\partial^\beta/\partial t^\beta$ is the Riemann–Liouville derivative of order $0 < \beta \leqslant 1$ and $\partial^\alpha/\partial |x_j|^\alpha$ is the Riesz derivative of order $0 < \alpha \leqslant 2$ operating on the space-symmetrical coordinate $|x_j|$ [26]. Bounding the intervals of the two exponents allows the correct probabilistic interpretation of the field as a positive scalar.

Under these circumstances (9) may be extrapolated to

$$\left[\left(\frac{\partial^{2\beta}}{\partial (t^0)^{2\beta}} - \nabla_0^{2\alpha}\right) + (m_0^0)^2\right]\varphi(\overrightarrow{r^0}, t) = 0 \tag{24}$$

Proceeding by analogy with (6), we perform the generalized operator substitution:

$$\begin{aligned} &E^\beta \to \mathrm{i}\frac{\partial^\beta}{\partial (t^0)^\beta} \\ &p_j^\alpha \to -\mathrm{i}\frac{\partial^\alpha}{\partial |x_j^0|^\alpha} \end{aligned} \tag{25}$$

which yields the fractional Klein–Gordon equation

$$[-E^{2\beta} + \vec{p}^{2\alpha} + (m_0^0)^2] = \varphi(\overrightarrow{r^0}, t) = 0 \tag{26}$$

where

$$\overrightarrow{p^\alpha} = \{p_j^\alpha\} \tag{27}$$

The resulting dispersion relation

$$E^{2\beta} = \vec{p}^{2\alpha} + (m_0^0)^2 \tag{28}$$

is an obvious generalization of the ordinary relativistic dispersion corresponding to $\alpha = \beta = 1$.

6. Scaling behavior of the effective Planck mass

We are now in a position to join the formalism outlined in (24)–(28) with the expression for the fractional Green function (19). The exponent f (defined inside the interval $(3, 4)$ for three-dimensional space, i.e. $3 < f < 4$) can be mapped to the exponent α (defined inside (0,1)) by using the substitution

$$\alpha = 4 - f \tag{29}$$

Analysis of dimensionless units employed in (25) and (28) yields the following set of correspondence relations

$$\begin{aligned} &M^0 \to E^\beta \to \frac{\partial^\beta}{\partial (t^0)^\beta} \to (t^0)^{-\beta} \\ &M^0 \to p^\alpha \to E^\beta \\ &p_j^\alpha \to \frac{\partial^\alpha}{\partial |x_j^0|^\alpha} \to (x_j^0)^{-\alpha} \to (r^0)^{-\alpha} \end{aligned} \tag{30}$$

As a result, the normalized coordinate scales as

$$r^0 = (r^0)_3 = (M^0)^{-\frac{1}{\alpha}} \tag{31}$$

which is the fractional generalization of the Compton wavelength (2) in the classical limit $\alpha \to 1$. Substituting (31) in (21) leads to a power-law scaling of the effective Planck mass assuming the form

$$M^0_{\mathrm{eff,Pl}} = (M^0_{\mathrm{Pl}})^{\lambda(\alpha)} \tag{32}$$

with

$$\lambda(\alpha) = \frac{4\alpha}{\alpha + 3} \tag{33}$$

This is the main result of the paper. Two asymptotic cases exist:

(1) The approach to classical limit $\alpha \to 1$ recovers the Planck mass as $\lambda(1) \to 1$ and $M_{\mathrm{eff,Pl}} \to M_{\mathrm{Pl}}$.
(2) As $\alpha \to 0$, $\frac{M_{\mathrm{eff,Pl}}}{M_0} \to 1$. Since the choice of mass scale M_0 is arbitrary, the effective Planck mass can take any arbitrary real value in $\{R_+\}$. This finding is consistent with the physics of critical behavior near transition points where there is a manifest loss of length scale and field correlations acquire an infinite range.

7. Scaling behavior of the effective gravity constant

Examination of (32) indicates that the effective gravity constant defined by

$$G_{\mathrm{eff}}(M_0, \alpha) = (M_{\mathrm{eff,Pl}})^{-2} = (M_{\mathrm{Pl}})^{\frac{-8\alpha}{\alpha+3}}(M_0)^{\frac{6(\alpha-1)}{\alpha+3}} \tag{34}$$

is no longer a universal constant, but a coupling strength depending on the dual choice of mass scale M_0 and exponent α. When $\alpha \to 1$, one recovers Newton's constant as

$$\underset{\alpha\to 1}{G_{\mathrm{eff}}} = G = (M_{\mathrm{Pl}})^{-2} \tag{35}$$

Fig. 1 shows the variation of $G_{\mathrm{eff}}(M_0, \alpha)$ as function of M_0 for a fixed α, namely $\alpha = 0.236$ (cf. next section). It is seen that gravity becomes an asymptotically free theory relative to the mass scale M_0. This is consistent with the behavior of non-abelian gauge theories at high energies [21,24].

The variation of $G_{\mathrm{eff}}(M_0, \alpha)$ as a function of α at a fixed mass scale M_0 is illustrated in Fig. 2 where M_0 was chosen to coincide with proton mass (cf. next section). It is seen that gravity behaves as a non-asymptotically free field theory for $\alpha \to 0$, which may have significant implications for the gravitational stability of bound states in regions of the α-space where the effective coupling is large.

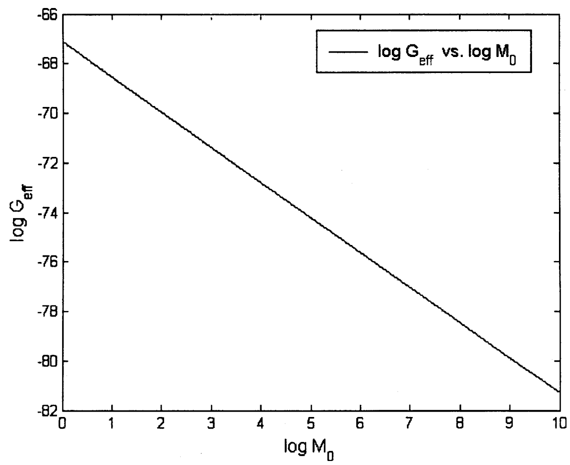

Fig. 1.

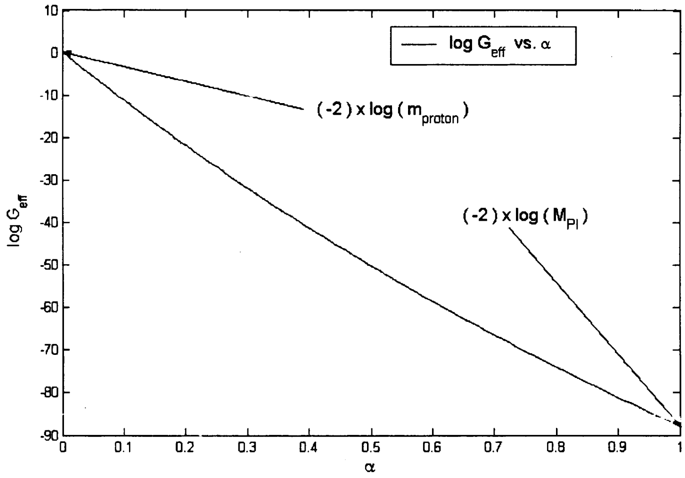

Fig. 2.

8. Connection to the geometry of Cantorian space–time

In what follows we use (34) to lower the numerical value of the effective Planck mass and suggest a simple solution to the gauge hierarchy problem. As pointed out in [18], exponent α is found to be dependent on the geometrical attributes of Cantorian space–time via

$$\alpha = \langle d_c \rangle - 4 = \phi^3 = 0.236068 \tag{36}$$

in which $\langle d_c \rangle$ is the expectation value for the Haussdorf dimension in E^∞ theory and $\phi = \frac{\sqrt{5}-1}{2}$ is the golden mean [17,23]. If, using a standard procedure, we choose the reference mass to coincide with the proton mass ($M_0 = m_{\text{proton}}$) [5,24], (34) is solved by

$$M_{\text{eff,Pl}} = 353.3 \text{ TeV}$$

We conclude that $M_{\text{eff,Pl}}$ has the same order of magnitude as the upper limit of the Cohen–Kaplan fundamental scale of gravity, placed near 100 TeV. This scale is considered to set the threshold of validity for any quantum-field theoretical description of nature [27,28].

9. Concluding remarks

A novel strategy for solving the gauge hierarchy problem has been presented. It is based on using fractal operators in the Helmholtz equation of classical field theory. We have found that:

(a) Deviation from the inverse-square gravity occurs naturally on high-energy scales.
(b) The effective gravity coupling depends on both the mass scale M_0 and the order of fractional differentiation in space (α).
(c) The effective Planck mass may be lowered in the Cohen–Kaplan range by linking α to a key geometrical attribute of Cantorian space–time, the golden mean ϕ.

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Nonlinear behavior of the renormalization group flow and standard model parameters

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Abstract

The standard model for high-energy physics (SM) describes fundamental interactions between subatomic particles down to a distance scale on the order of 10^{-18} m. Despite its widespread acceptance, a consistent and comprehensive understanding of SM parameters is missing. Starting from a less conventional standpoint, our work suggests that the spectrum of particle masses, gauge couplings and fermion mixing angles may be derived from the chaotic regime of the renormalization group flow. In particular, we argue that the observed hierarchies of standard model parameters amount to a series of scaling ratios depending on the Feigenbaum constant. Leading order predictions are shown to agree well with experimental data.

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Keywords: Renormalization group flow; Period doubling bifurcations; Feigenbaum scaling; Standard model parameters

1. Introduction

The generation structure of quarks and leptons stands out as one of the most intriguing puzzles of the standard model for particle physics (SM). The conventional formulation of the SM requires 19 free input parameters, among which 12 can be expressed in terms of empirical mass eigenvalues [1]. In addition, there is a set of four inputs determined by the so-called Cabibbo–Kobayashi–Maskawa (CKM) matrix whose structure includes three quark-mixing angles and one CP phase [18,19]. The remaining three parameters are two gauge couplings (α_3, α_{em}) and the strong CP phase. Recent experiments in neutrino physics have confirmed the existence of neutrino oscillations and masses and have subsequently triggered a host of challenging questions [2–4]. There is a large body of proposed extensions of SM, each of them attempting to resolve some unsatisfactory aspects of the theory while introducing new unknowns. In contrast with the line of thought pursued by

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these models, our work suggests that the spectrum of particle masses, gauge couplings and fermion mixing angles may be derived from the *chaotic behavior of the renormalization group* (*RG*) flow. Although predictions are found to match reasonably well experimental observations, we caution that our results are entirely preliminary and a concurrent analysis is needed to confirm or disprove their validity.

The standard procedure for investigating the high-energy domain of any effective field theory is to start from the underlying RG flow equations, identify its fixed points and analyze the asymptotic flow of coupling parameters in the basin of attraction of these points [5]. Taking an alternative approach, we treat the RG flow equation as a generic iterated mapping and evaluate its chaotic regime after a large number of iteration cycles. We conclude that the observed hierarchy of SM parameters amounts to a series of scaling ratios depending on the Feigenbaum constant [27]. Since fermion mass scaling ratios and mixing matrices can be parameterized in terms of the Cabibbo angle [6–8,28], this result supplies a natural connection between the Cabibbo angle and the Feigenbaum constant. Moreover, it is found that the model can accommodate hypothetical generations of both heavy and ultra-light fermions that are expected to emerge beyond the energy range of SM. A representative example in this regard is the fourth SM family neutrino whose detection is anticipated at future linear colliders [9].

The paper is organized in the following way: Section 2 outlines the background of the RG flow equation and derives the asymptotic link between the beta-function and Feigenbaum scaling for a generic effective field theory. The emergence of a hierarchical pattern of observables based on this link is elaborated upon in Section 3 with specifics on SM hierarchies detailed in Section 4. The last three sections include a brief presentation of future extensions, open questions and concluding remarks.

2. Beta-function and Feigenbaum scaling

Following the framework of RG transformations, all physical observables of an effective field theory can be formulated in terms of a finite number of renormalized couplings [10]. These are defined at an arbitrary mass scale μ referred to as a ‘‘subtraction point’’ or ‘‘sliding scale’’. One key result of RG is that any change in the renormalized correlation functions in response to a variation in μ must be compensated by a corresponding change in the renormalized couplings. The outcome of this conjecture is contained in the so-called Callan–Symanzik equation, which reflects how all observables of the theory change (or ‘‘flow’’) with μ. Beta-function of the renormalization group flow is defined by the partial differential equation

$$\beta[g(\mu)] \doteq \mu \frac{\partial g(\mu)}{\partial \mu} \tag{1}$$

The zeroes of the beta-function, generically called fixed points, are of particular interest in the theory of RG flows. Knowledge of the fixed points enables the study of high and low-energy domains of the effective field theory [10,11].

We proceed from these preliminary considerations by introducing the following set of working assumptions:

1. The effective field theory contains a single coupling parameter $g = g(\mu)$.
2. The asymptotic flow of the coupling parameter toward the fixed point g^* reflects the approach to the high-energy domain of field theory.
3. The phase transition associated with the flow $g(\mu) \to g^*$ is an infinite-order phase transition.

The last two assumptions may be linked to framework of conformal field theories, which are considered well suited for the description of high-energy physics [12–14]. A remarkable feature of infinite-order phase transitions is that the correlation length ξ has an essential singularity at the critical coupling g^* given by

$$\xi \sim \exp(A|g - g^*|^{-\sigma}) \tag{2}$$

in which σ is a critical exponent and A a constant. Such behavior develops when the coupling parameter has a vanishing mass dimension at g^* and the beta-function may be represented as a quadratic function of g [12–14]

$$\mu\frac{\partial g}{\partial\mu} = cg^2 + O(g^3) \tag{3}$$

where c is a real-valued coefficient. The discrete analogue of (3) reads

$$g_{n+1} - g_n = \frac{c\Delta\mu}{\mu}g_n^2 + O(g_n^3) \tag{4}$$

Here n is the iteration index and the subtraction point increment $\Delta\mu$ represents a scalar fixed by resolution requirements. Any realistic description of the RG flow in the high-energy domain must take into account statistical fluctuations stemming from the uncertainty principle. Because large fluctuations and non-equilibrium microscopic processes dominate the physics on short time scales, the temporal resolution $\Delta t_n \doteq t_{n+1} - t_n$ is expected to vary as the inverse of time measurement, i.e. $\Delta t_n \sim t_n^{-1}$. It follows that the dimensionless subtraction point entering (4) and defined as $\tilde{\mu} \doteq \mu/c\Delta\mu$ acts as an autonomous control parameter. Following the onset of chaos in quadratic maps through period doubling bifurcations, it can be shown that the transition from a period 2^n super stable orbit to a period 2^{n+1} super stable orbit occurs for a geometrically spaced series of control parameters given by [15]

$$\tilde{\mu}_n - \tilde{\mu}_\infty \sim \delta_2^{-n} \tag{5}$$

where $2^n \gg 1$ and where $\delta_2 \doteq 4.669\ldots$ is the Feigenbaum constant for the quadratic map. From the previous discussion it can be inferred that $\tilde{\mu}_\infty$ represents the fixed point of the $\tilde{\mu}_n$ series whose generic term is defined via

$$\tilde{\mu}_n \doteq \frac{\mu}{c\Delta\mu_n} \sim \frac{\mu}{c}t_n^{-1} \tag{6}$$

It is important to emphasize that (5) is frame-independent, in the sense that its form is not affected by changing the subtraction point and its limit $\tilde{\mu}'_n = s\tilde{\mu}_n$, $\tilde{\mu}'_\infty = s\tilde{\mu}_\infty$ with $s \in \{R\}$.

To streamline the derivation and without losing generality, we further assume that a plausible boundary condition in (5) is $\tilde{\mu}_\infty \approx 0$. This ansatz may be justified by considering that the RG flow develops over sufficiently large times ($t_n \gg 0$).

The emergence of scaling (5) points out to an important result regarding the asymptotic form of the beta-function. According to the guiding prescription of RG analysis, the evolution of the beta-function may be studied through a sequence of renormalization steps consisting of iterated composition and rescaling operations [15,16]. Let $\tilde{\beta}(g)$ designate the universal Feigenbaum–Cvitanovic function that satisfies the so-called renormalization equation

$$\tilde{\beta}(g) \doteq -\alpha\tilde{\beta}\left(\tilde{\beta}\left(\frac{g}{\alpha}\right)\right) \tag{7}$$

in which $\alpha \doteq 2.5029\ldots$ After a large number of iteration cycles ($2^n \gg 1$), the renormalized beta-function $\tilde{\beta}_n(g) \doteq (-\alpha)^n\beta^{(2^n)}\left(\frac{g}{\alpha^n}, \tilde{\mu}_n\right)$ approaches $\tilde{\beta}(g)$ according to [15,16]

$$\tilde{\beta}(g) = \lim_{n\to\infty}(-\alpha)^n\beta^{(2^n)}\left(\frac{g}{\alpha^n}, \tilde{\mu}_\infty\right) \tag{8}$$

The renormalized beta-function obeys the recursive relation

$$\tilde{\beta}_{n-1}(g) \doteq -\alpha\tilde{\beta}_n\left(\tilde{\beta}_n\left(\frac{g}{\alpha}\right)\right) \tag{9}$$

such that

$$\tilde{\beta}_n(g) - \tilde{\beta}(g) \sim \delta_2^{-n}h(g) \tag{10}$$

where $h(g)$ is an analytic function. Moreover, since our focus is the coupling flow in the immediate neighborhood of g^*, where $(g - g^*) \sim O(\varepsilon)$, we may reasonably assume that $\tilde{\beta}(g) \sim O(\varepsilon)$. We arrive at

$$\tilde{\beta}_n(g) \sim \delta_2^{-n}h(g) \tag{11}$$

The above power-law behavior reveals the asymptotic connection between the renormalized beta-function, on the one hand, and Feigenbaum constant on the other. Next sections explore the impact of this result on key observables describing a typical field theoretic framework such as the electroweak model or QCD.

3. Hierarchical pattern of observables

Let Ω be a generic observable of the effective field theory such as mass, gauge coupling or mixing angle. Assuming that the dimension of Ω is d_0, that is

$$[\Omega] = [\mu]^{d_0} \tag{12}$$

we may write, by dimensional analysis [17]

$$\Omega(\mu, g(\mu)) = \mu^{d_0} f_{d_0}(g(\mu)) \tag{13}$$

Constraining the function $f_{d_0}(g(\mu))$ to be independent of the subtraction point yields

$$\frac{\mathrm{d}\Omega(\mu, g(\mu))}{\mathrm{d}\mu} = 0 \to f_{d_0}(g(\mu)) \sim \exp\left(-d_0 \int_{g^*}^{g(\mu)} \frac{\mathrm{d}g}{\tilde{\beta}(g)}\right) \tag{14}$$

On account of the RG interpretation previously developed, the dimensionless form of (13) may be written as

$$\Omega_n(\tilde{\mu}_n, g(\tilde{\mu}_n)) \sim \tilde{\mu}_n^{d_0} \exp[-\delta_2^n d_0 F(g(\tilde{\mu}_n), g^*)] \tag{15}$$

with

$$F(g(\tilde{\mu}_n), g^*) \doteq \int_{g^*}^{g(\tilde{\mu}_n)} \frac{\mathrm{d}g}{h(g)} \tag{16}$$

The integral (16) may be approximated around $\tilde{\mu}_\infty \approx 0$ as

$$F(g(\tilde{\mu}_n), g^*) \approx \frac{g(\tilde{\mu}_n) - g^*}{h(g^*)} \approx \tilde{\mu}_n \frac{\left[\frac{\partial g}{\partial \tilde{\mu}_n}(0)\right]}{h(g^*)} = \delta_2^{-n} \frac{\left[\frac{\partial g}{\partial \tilde{\mu}_n}(0)\right]}{h(g^*)} \tag{17}$$

which implies that, for two arbitrary iteration indices,

$$\frac{\Omega_n(\tilde{\mu}_n, g(\tilde{\mu}_n))}{\Omega_m(\tilde{\mu}_m, g(\tilde{\mu}_m))} \sim \delta_2^{(m-n)d_0} \tag{18}$$

We end this section by noting that $d_0 = 1$ if the observable (13) refers to a mass parameter and $d_0 \sim \mathrm{O}(\varepsilon)$ if it refers to a gauge coupling or a mixing angle. The latter property is a direct consequence of (2) which implies that coupling charges behave as marginal parameters in the immediate neighborhood of g^* [12,13]. In this case it is reasonable to assume that, on a first-order basis, the index difference $(m-n)d_0$ for $m, n \gg 1$ may be rounded off to the closest integer.

4. Scaling hierarchies of standard model parameters

A remarkable yet unexplained property of SM parameters is that they appear to be organized in a hierarchical fashion. The scaling ratio of two parameters in the hierarchy depends on integer powers of the Cabibbo angle whose experimental best-fit value is $\theta_C = 12.9$–$13°$ [23]. It is customary to work with the Cabibbo angle in the equivalent trigonometric form, that is, $\lambda \doteq \sin\theta_C = 0.223$–$0.225$. Let us assume that the set of charged lepton and current-quark masses, evaluated at an arbitrary energy scale, are denoted by the vector M_l and matrix M_q, respectively

$$M_l \doteq [m_e \quad m_\mu \quad m_\tau] \quad M_q \doteq \begin{bmatrix} m_u & m_d \\ m_c & m_s \\ m_t & m_b \end{bmatrix} \tag{19}$$

The explicit set of scaling ratios in (19) is given by [6–8,21,26]:

$$\begin{aligned} &\frac{m_e}{m_\mu} \sim \lambda^4 \quad \frac{m_\mu}{m_\tau} \sim \lambda^2 \\ &\frac{m_c}{m_t} \sim \lambda^4 \quad \frac{m_s}{m_b} \sim \lambda^2 \\ &\frac{m_u}{m_t} \sim \lambda^8 \quad \frac{m_d}{m_b} \sim \lambda^4 \end{aligned} \tag{20}$$

We note that the pattern of charged fermion masses depends on integer powers of λ^2. Here, quark masses are arranged in two columns each involving three independent flavors, namely (u, c, t) and (d, s, b).

It is pertinent to bring up at this point the issue of fermion mixing and its parameterization. As it is known, in the SM quark mass eigenstates are different from their weak eigenstates partners and the CKM matrix, denoted by V_{CKM}, relates these two bases by operating on the $(-1/3)$ mass states (d, s, b) [18,19]:

$$\begin{bmatrix} d' \\ s' \\ b' \end{bmatrix} = V_{\mathrm{CKM}} \begin{bmatrix} d \\ s \\ b \end{bmatrix} \tag{21}$$

In terms of individual mixing components, we have

$$V_{\mathrm{CKM}} \doteq \begin{bmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{bmatrix} \tag{22}$$

Unlike (20), the CKM matrix expressed using the so-called Wolfenstein parameterization [20] is approximated to the leading order by entries dependent on integer powers of λ [1]

$$V_{\mathrm{CKM}} \approx \begin{vmatrix} 1 & \lambda & \lambda^3 \\ -\lambda & 1 & \lambda^2 \\ \lambda^3 & -\lambda^2 & 1 \end{vmatrix} \tag{23}$$

A similar matrix structure may be assigned to the recently discussed set of operators describing mixing in the lepton sector [21,22]. Specifically, if the neutrino mass matrix m_ν and the charged lepton mass matrix m_l are diagonalized through the following transformations

$$\begin{aligned} m_\nu &\doteq U_\nu m_\nu^{\mathrm{diag}} U_\nu^T \\ m_l &\doteq U_L m_l^{\mathrm{diag}} U_R^+ \end{aligned} \tag{24}$$

then it can be shown that neutrino mixing, defined by the so-called Pontercovo–Maki–Nakagawa–Sakata (PMNS) matrix, may be represented as

$$U_{\mathrm{PMNS}} = U_L^+ U_\nu \tag{25}$$

In one plausible scenario, one finds [21]

$$\begin{aligned} m_l m_l^+ &\sim m_\tau^2 \begin{vmatrix} \lambda^6 & \lambda^5 & \lambda^3 \\ \lambda^5 & \lambda^4 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{vmatrix} \\ U_L m_l^{\mathrm{diag}} U_L^T &\sim m_\tau \begin{vmatrix} \lambda^4 & \lambda^3 & \lambda^3 \\ \lambda^3 & \lambda^2 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{vmatrix} \end{aligned} \tag{26}$$

The standard parameterization of the PMNS matrix is formulated with the help of three mixing angles $(\theta_{12}, \theta_{23}, \theta_{13})$. According to the above scenario, we have

$$\begin{aligned} \sin\theta_{12} &= \lambda \\ \sin\theta_{13} &= A\lambda^3 \\ \sin\theta_{23} &= B\lambda^2 \end{aligned} \tag{27}$$

where A, B are positive numbers of order unity.

[1] To simplify the argument, CP-violating phases are neglected here.

Finally, it is instructive to recall that SM coupling charges and weak boson masses satisfy the following scaling pattern [23]

$$\begin{aligned} &\left(\frac{e}{g_2}\right)^2 \sim \lambda_\mathrm{W} \quad \left(\frac{e}{g_{3s}}\right)^2 \sim \lambda_\mathrm{W}^2 \\ &\left(\frac{M_\mathrm{W}}{M_Z}\right)^2 \sim 1 - \lambda_\mathrm{W} \end{aligned} \tag{28}$$

Here, $e^2 \doteq 4\pi\alpha_\mathrm{em}$, $g_2^2 \doteq 4\pi\alpha_2$, $g_{3s}^2 \doteq 4\pi\alpha_{3s}$ stand for the electromagnetic, weak and strong coupling charges and λ_W is the "sine" squared of the Weinberg angle, whose magnitude is nearly identical to the nominal value of the Cabibbo angle ($\lambda_\mathrm{W} \doteq \sin^2\theta_\mathrm{W} = 0.229$) [10,23].

As stated at the beginning of this section, relationships (19)–(28) provide ample analytical evidence that SM parameters display a hierarchical dependence on the Cabibbo angle. This observation is consistent with (18) and strongly suggests a direct connection between λ and δ_2. In fact

$$\lambda \sim \delta_2^{-1} = 0.214\ldots \tag{29}$$

which leads one to conclude that the Feigenbaum constant for the quadratic map plays a central role in the observed patterns of particle masses, gauge couplings and fermion mixing angles.[2]

5. Future extensions

As it is known, the SM does not fix the number of fermion families. For example, current data allow for additional generations of leptons and quarks if the mass of the fourth family neutrino is larger than $M_Z/2$ [9]. At the other end of the energy scale, various studies on neutrinoless double beta-decay processes point to a spectrum of ultra-light neutrinos with masses well below the eV threshold [25]. As the ladder-like pattern of SM parameters encoded in (18) and (19) is not bounded by fixed limits on the index difference $(n - m)d_0$, one may infer that new fermion generations arise beyond what is known today. The object of this section is to formulate first-order predictions on the hypothetical ultra-light and super-heavy fermion masses that may be observed in future experiments. The most straightforward extrapolation of (20) on account of (29) gives

$$\begin{aligned} &m_{l4} \sim m_{\nu_e}\delta_2^{-2} < 4.6 \times 10^{-2}\ \mathrm{eV} \\ &m_{l5} \sim m_\tau\delta_2^{2} = 38.76\ \mathrm{GeV} \\ &m_{q4} \sim m_u\delta_2^{-2} = 0.107\ \mathrm{eV} \\ &m_{q5} \sim m_t\delta_2^{2} = 3.95\ \mathrm{TeV} \end{aligned} \tag{30}$$

Here, $l4$, $q4$ ($l5$, $q5$) denote the ultra-light (super-heavy) families of leptons and quarks, respectively, whereas $m_{\nu_e}(< 1\ \mathrm{eV})$, m_τ, m_u, m_t are best-fit fermion masses evaluated at the Z boson scale [9,26]. We find that these numbers agree well with predictions derived from the models developed in [9,25].

6. Open questions

The primary goal of this work was to present arguments that support an unexpected connection between the Feigenbaum scaling and RG, on the one hand, and SM hierarchies, on the other. Needles to say, our study does not provide a rigorous and comprehensive account of the physics underpinning the generation structure of the SM. Many questions remain open. Their satisfactory resolution requires a more extensive and refined plan of attack as well as a wealth of currently unavailable experimental data. Although a complete list of questions is not a practical option, we believe that among the most pressing issues that need to be dealt with are the following:

[2] A similar scenario is analyzed in [24] where mass generation in the lepton sector arises from the dissipative chaotic dynamics of the basic weak boson-fermion system.

1. What explains the small numerical difference $\lambda - \delta_2^{-1} \approx 9 \times 10^{-3}$? Are contributions related to higher nonlinear terms in (3) and (4) relevant to this context?
2. Why is the mass hierarchy dependent on integer powers of δ_2^{-2} whereas the gauge coupling and mixing angle hierarchy depend on integer powers of δ_2^{-1}?
3. What mechanism is responsible for maintaining the parameter hierarchy in the transition from the high-energy domain of field theory to the low-energy domain of the SM?

7. Concluding remarks

We have suggested that the chaotic behavior of the RG flow offers valuable insights into the generation puzzle of the SM. In particular, it was argued that the observed hierarchies of standard model parameters amount to a series of scaling ratios depending on the Feigenbaum constant. A direct link was found between this constant and the Cabibbo angle. Future generations of "would-be" heavy and ultra-light fermions may be extrapolated using this dynamical model.

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On the asymptotic transition to complexity in quantum chromodynamics

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Abstract

Quantum chromodynamics (QCD) is a renormalizable gauge theory that successfully describes the fundamental interaction of quarks and gluons. The rich dynamical content of QCD is manifest, for example, in the spectroscopy of complex hadrons or the emergence of quark–gluon plasma. There is a fair amount of uncertainty regarding the behavior of perturbative QCD in the infrared and far ultraviolet regions. Our work explores these two domains of QCD using non-linear dynamics and complexity theory. We find that local bifurcations of the renormalization flow destabilize asymptotic freedom and induce a steady transition to chaos in the far ultraviolet limit. We also conjecture that, in the infrared region, dissipative non-linearity of the renormalization flow supplies a natural mechanism for confinement.

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Keywords: Complex dynamics; Transition to chaos; Quantum chromodynamics; Quark–gluon plasma

1. Introduction and motivation

As a building block of the Standard Model for particle physics, QCD is a successful gauge theory describing the coupling of quarks and gluons [1–4]. It has several defining features, namely: (a) asymptotic freedom (the interaction becomes weaker at short distances and it can be determined from perturbation theory), (b) around 200 MeV, confinement sets in and the particle spectrum consists exclusively of color neutral states, (c) QCD exhibits spontaneous chiral symmetry breaking due to non-vanishing quark masses [1–4], (d) at high temperature or high density, QCD is conjectured to sustain phase transitions leading to quark–gluon plasma and the restoration of chiral symmetry [4]. Due to asymptotic freedom, perturbative QCD is reasonably effective in the high-energy limit but fails to provide accurate predictions in the infrared limit, where the theory becomes strongly coupled [1–5]. The infrared regime of QCD is a typical example where non-perturbative methods

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become compelling. Since closed-form solutions of field theory are, in general, difficult to extract and manage, lattice-based computations and numerical approximations are among the most frequently used techniques for investigation [2,3]. Less developed are methods based on non-linear analysis and dynamical systems theory, whereby knowledge of explicit solutions is no longer critical. From this standpoint, it can be stated that non-linear dynamics offers an attractive theoretical laboratory for probing the asymptotic dynamics of QCD. With regard to field theory in general, this is also true near any boundary of the stability region where randomness becomes the driving factor [6] and the emergence of bifurcations and complex behavior is a likely occurrence. It is in this region where traditional procedures are questionable and one usually appeals instead to alternative methods such as the ones provided by the renormalization group (RG) [7]. Starting from these considerations, our goal is to develop a first-order analysis of the RG flow near the boundary of the stability region. The sustained contribution of perturbations to the RG flow is modeled as follows: (a) since QCD is asymptotically free, we assume that perturbations develop progressively but smoothly in the ultraviolet region, (b) in contrast, because QCD becomes strongly coupled in the infrared, we assume that perturbations are best modeled here as random fluctuations of Levy type. We caution that our work has an introductory nature and does not claim to provide a comprehensive and rigorous coverage of the topic. As the contribution of fluctuations and non-linearities becomes increasingly predominant in the asymptotic regime of QCD, a complete analysis needs to carefully account for a variety of factors that are deliberately left out in our derivation.

The paper is organized according to the following plan: Section 2 examines the QCD dynamics in the far ultraviolet region; the impact of Levy noise on the mechanism of infrared confinement is outlined in Section 3. The last section contains a brief summary of results. Appendix A includes a condensed presentation of RG equations in the context of perturbative QCD.

2. QCD dynamics in the far ultraviolet region

2.1. Perturbed RG flow equations

We start from the RG equations for coupling strength and quark masses [8]

$$\begin{aligned} \frac{d\alpha_s}{dt} &\approx -b_0(n)\alpha_s^2 - b_1(n)\alpha_s^3 \\ \frac{dm}{dt} &\approx -m\left[c_0(n)\alpha_s + c_1(n)\alpha_s^2 + c_2(n)\alpha_s^3\right] + \mathrm{NP} \end{aligned} \tag{1}$$

Here, n stands for the number of quark flavors and

$$t = \ln\left(\frac{\Lambda}{\Lambda_0}\right) \tag{2}$$

represents the sliding scale, where the momentum cutoff Λ is normalized to an arbitrary reference value Λ_0 such as the strong interaction scale ($\Lambda_0 \cong 220$ MeV). The non-perturbative term in the mass flow is denoted by NP and is typically presumed to vanish faster than any power of the coupling [9]. In the presence of generic perturbations (1) becomes

$$\begin{aligned} \frac{d\alpha_s}{dt} &\approx -b_0(n)\alpha_s^2 - b_1(n)\alpha_s^3 + \Phi(\alpha_s) \\ \frac{dm}{dt} &\approx -m\left[c_0(n)\alpha_s + c_1(n)\alpha_s^2 + c_2(n)\alpha_s^3\right] + \Psi(m) \end{aligned} \tag{3}$$

Let us assume that the two additive contributions may be expanded in power series of a small parameter that defines the perturbation amplitude ($\varepsilon \ll 1$)

$$\begin{aligned} \Phi(\alpha_s) &= \Phi_0(\alpha_s) + \varepsilon\Phi_1(\alpha_s) + \varepsilon^2\Phi_2(\alpha_s) + \cdots \\ \Psi(m) &= \Psi_0(m) + \varepsilon\Psi_1(m) + \varepsilon^2\Psi_2(m) + \cdots \end{aligned} \tag{4}$$

For simplicity, we take

$$\begin{aligned} &\Phi_n(\alpha_s) = \Psi_n(m) = 0, \quad \text{if } n \neq 1 \\ &\Phi_1(\alpha_s) = \alpha_s^2, \quad \Psi_1(m) = -m \end{aligned} \tag{5}$$

(3) is thereby well approximated by

$$\frac{d\alpha_s}{dt} \approx -[b_0(n) + \varepsilon]\alpha_s^2 - b_1(n)\alpha_s^3 \equiv -b_\varepsilon(n)\alpha_s^2 - b_1(n)\alpha_s^3 \tag{6a}$$

$$\frac{dm}{dt} \approx -m\left[\varepsilon + c_0(n)\alpha_s + c_1(n)\alpha_s^2 + c_2(n)\alpha_s^3\right] \tag{6b}$$

in which

$$b_\varepsilon(n) = b_0(n) + \varepsilon \tag{7}$$

2.2. Linear stability analysis

Apart from the trivial solution represented by the fixed point $FP_0 = [\alpha_s^* = 0, m^* = 0]$, the non-trivial fixed point of (6a) is given by $FP = [\alpha_s^* = -(b_\varepsilon/b_1), m^* = 0]$. Linearizing around FP_0, we find that the Lyapunov exponent is vanishing regardless of the numerical value of coefficients b_0, b_1 and, implicitly, regardless of the number of flavors n. On the other hand, the Lyapunov exponents corresponding to FP are [10,11]

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} \tag{8}$$

in which

$$\begin{aligned} \tau &= -\frac{b_\varepsilon^2}{b_1} + \varepsilon + \frac{c_0 b_\varepsilon}{b_1} - \frac{c_1 b_\varepsilon^2}{b_1^2} + \frac{c_2 b_\varepsilon^3}{b_1^3} \\ \Delta &= -\frac{b_\varepsilon^2}{b_1}\left(\varepsilon + \frac{c_0 b_\varepsilon}{b_1} - \frac{c_1 b_\varepsilon^2}{b_1^2} + \frac{c_2 b_\varepsilon^3}{b_1^3}\right) \end{aligned} \tag{9a}$$

To streamline the analysis, we next assume that all terms and factors dependent on the "c" coefficients are negligible. Following the general guidelines of non-linear analysis, we are interested in the so-called borderline cases (i.e. centers, non-isolated fixed points, degenerate nodes and stars). These are determined by the numerical value of the characteristic parameter [11]

$$R = \tau^2 - 4\Delta \tag{9b}$$

Fig. 1 graphs the variation of R as a function of n as R approaches zero (red = 0.1, blue = 0.05, black = 0.001). It confirms that the dynamics of the RG flow becomes borderline as the number of quark flavors approaches $n_{cr} = 16$ and QCD reaches the point of losing its asymptotic freedom [12].

2.3. Bifurcation of the fixed point

We now wish to study the behavior of the non-trivial FP under the influence of a steadily increasing perturbation whose amplitude augments the noise terms previously considered ($\varepsilon \ll 1$). The origin of this perturbation may be related to thermal fluctuations (if the analysis is carried out at a high temperature setting) or to the presence of a large number of high-order diagrams associated with the ultraviolet limit. To this end, let us add an infinite series of terms to the coupling flow equation (6a), that is

$$\frac{d\alpha_s}{dt} \approx -b_\varepsilon(n)\alpha_s^2 - b_1(n)\alpha_s^3 + \sum_{i=0}^{\infty}\kappa_i(n)\alpha_s^i \tag{10}$$

We assume next that only the first two terms in the series (10) are non-vanishing and weakly dependent on n. Following the notation of [10], we obtain

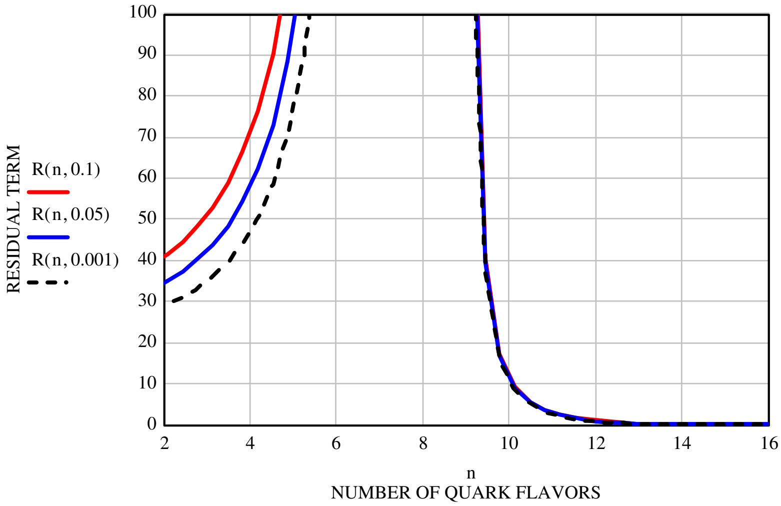

Fig. 1. Characteristic parameter ξ versus the number of quark flavors n.

$$\frac{d\alpha_s}{dt} \approx \kappa_1 + \alpha_s(\kappa_2 + \Lambda_1) + \Lambda_2\alpha_s^2 + \Lambda_3\alpha_s^3 \tag{11}$$

Here, $\Lambda_{1,2,3}$ denote the so-called Lyapunov values which are, respectively, given by $\Lambda_1 = -\frac{6b_\varepsilon^2}{b_1}$, $\Lambda_2 = b_\varepsilon$, $\Lambda_3 = -b_1$. The FP is stable since $\Lambda_3 < 0$. The set of scalars $\kappa_{1,2}$ denote the governing parameters and measure the deviation of an arbitrary point in parameter space from origin ($\kappa_1 = 0, \kappa_2 = 0$). Under these circumstances, the bifurcation curve has a cusp profile and is represented by [10,11]

$$\kappa_1 = \frac{\pm 2\kappa_2\sqrt{|\kappa_2/(-b_1)|}}{3\sqrt{3}} + \cdots, \quad \kappa_2(-b_1) < 0 \tag{12}$$

Fig. 2 plots the variation of the Lyapunov values as a function of n. Fig. 3 shows the emergence of a cusp bifurcation in the (κ_1, κ_2) plane when b_1 is computed at $n = 6$ (see (A2)). Depending on the location of the

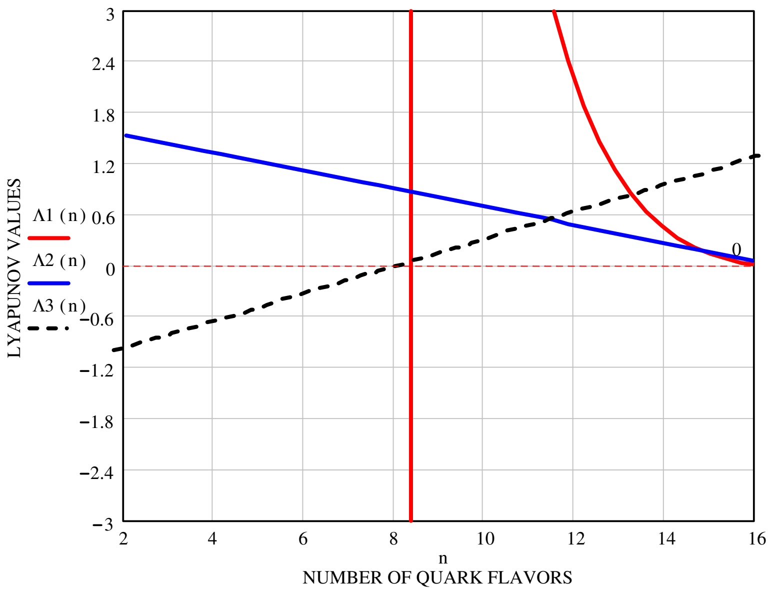

Fig. 2. Lyapunov values versus the number of quark flavors n.

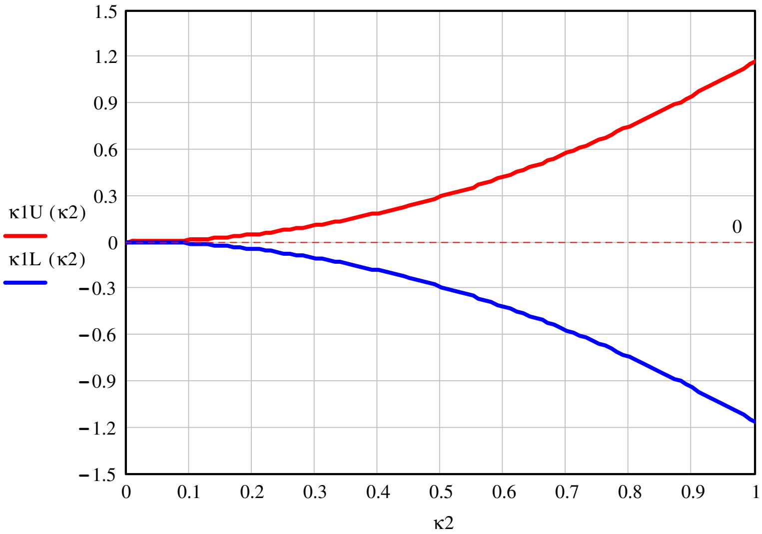

Fig. 3. Cusp bifurcation in the (κ_1, κ_2) plane.

governing parameters in the (κ_1, κ_2) plane, the stable FP stays unchanged or splits into two or three equilibria. It follows that, near the non-trivial FP, irregular behavior of the coupling flow is likely to develop through a progressive cascade of cusp bifurcations.

3. Levy noise as possible mechanism for confinement

As noted in the first section, the infrared limit is characterized by large fluctuations of the RG flow induced by the strong-coupling regime of QCD. To preserve maximum generality of our approach and using arguments related to the ubiquity of Levy flights in stochastic transport processes [13], we model these fluctuations with the help of the generalized Langevin equation [14]

$$\frac{d\alpha_s}{dt} = \eta(\alpha_s)\alpha_s + \Gamma_\alpha(t) \tag{13}$$

Here, $\Gamma_\alpha(t)$ represents α-stable Levy noise and the dissipative non-linearity $\eta(\alpha_s)$ is given by

$$\eta(\alpha_s) = -b_\varepsilon(n)\alpha_s - b_1(n)\alpha_s^2 \tag{14}$$

Under these conditions, the asymptotic probability distribution function for $\alpha_s \gg 1$ is given by [14]

$$p(r, \alpha_s) \propto \frac{1}{b_r|\alpha_s|^\sigma} \tag{15}$$

in which $b_r > 0$ stands for the rth order coefficient of (14) and

$$\sigma = 1 + r + \alpha \tag{16}$$

It follows that $\langle \alpha_s^2 \rangle$ stays finite if $r > r_{cr} = 2 - \alpha$, that is, $r > 0$ if $\alpha = 2$ and $r > 2$ if $\alpha = 0$. We conclude that the coupling flow driven by stable Levy noise remains confined if its expansion is taken at least to the second loop approximation. This ansatz suggests a plausible mechanism for confinement in the IR region of QCD: instead of reaching a regime of unbounded variations in interaction amplitude, higher order radiative corrections generated from $\eta(\alpha_s)$ dissipate the energy imparted by Levy noise. As a result, quarks and gluons form bounded states with a nearly constant average coupling strength.

It is instructive to note that this conjecture fits well various lattice studies and phenomenological theories of quark–antiquark $(q\bar{q})$ interaction, such as the Richardson or Cornell models [2,3]. For example, the Cornell model assumes that the long-range part of the static $q\bar{q}$ potential has the form

$$V(r) = br - \frac{a}{r} + V_0 \tag{17}$$

where a, b, V_0 are constants. The coefficient b is commonly referred to as the ''string tension'' by comparison with string theories of hadrons. The linear term of this potential (br) dominates the interaction at large distances where it models a color-flux tube of constant energy density.

It is also instructive to remark that, in a certain sense, the mechanism of confinement produced by Levy fluctuations is similar to the phenomenon of Anderson localization in which quantum waves become confined in random potentials [15].

We close this section with an evaluation on how statistical moments of coupling strengths and quark masses depend on the Levy parameter α. For this purpose, it is sufficient to solve (1) in closed-form and use the asymptotic probability distribution function (15) to determine the expectations and variances for coupling

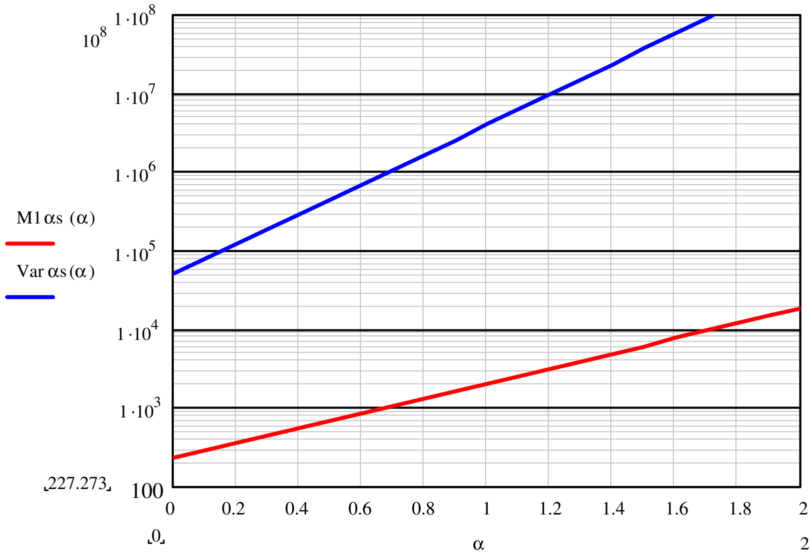

Fig. 4. QCD coupling strength and its variance versus α (log scale).

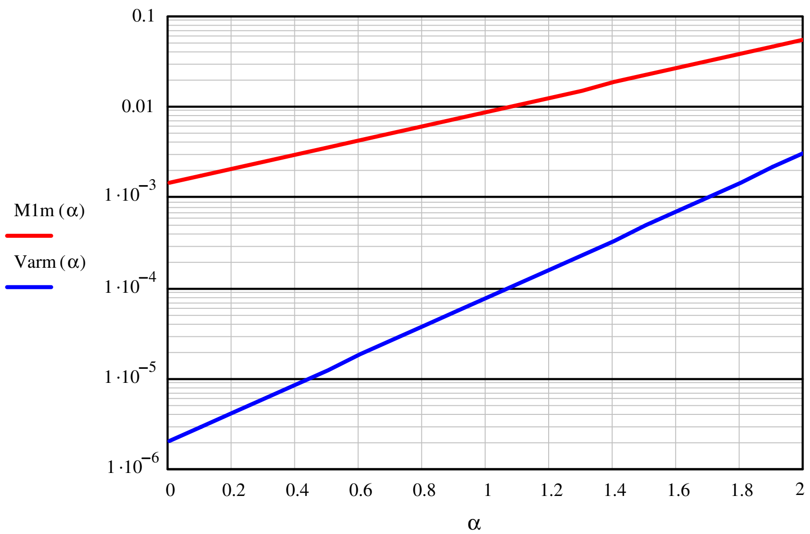

Fig. 5. Quark mass and its variance versus α (log scale).

strength and mass. Fig. 4 plots the expectation of the coupling strength ($M_1\alpha_s$) along with its variance ($\mathrm{Var}\,\alpha_s$) as functions of the Levy index α, whereas Fig. 5 plots the same behavior for the quark mass. The number of quark flavors is assumed to be $n = 6$ in both cases. It is seen that the increase of variances with the Levy parameter is significantly faster than the corresponding increase of expectation values.

4. Summary and discussion

Using the analytical tools provided by non-linear dynamics and complexity theory, we have examined the effect of the renormalization flow in the asymptotic regions of QCD. It was found that the steady addition of perturbations destabilizes asymptotic freedom and induces transition to chaos in the far ultraviolet limit. At the other end of the energy scale, dissipative non-linearity of the renormalization flow provides a plausible mechanism for confinement.

In Section 1, we pointed out the introductory nature of our treatment. QCD is a rich and complex theory that has the potential of exhibiting a large spectrum of behaviors (see, for instance [24,25]). It is apparent that the outcome of the set of coupled non-linear equations describing the RG flow depends strongly on how the model is formulated and how the boundary conditions are set. To be specific,

(1) There are two dependent control parameters of the RG flow: the momentum scale Λ (or, equivalently, the dimensional regularization parameter $\varepsilon = 4 - d$ [1,7]) and the number of fermion flavors $n = n(\Lambda) = n(t)$. Obviously, a simplified setting is to assume $n = n(\Lambda) = n(t)$ is a slowly varying function and carry the analysis with a single control parameter defining the energy scale at which the physics is probed.
(2) The dimensionality of the flow plays a critical role: a planar system of equations (such as the one for coupling and masses) does not lead to deterministic chaos. In contrast, the 3D system containing the flow of fields and mixing angles leads to a much richer spectrum of behaviors, including deterministic chaos.
(3) Addition of statistical perturbations leads to systems of coupled stochastic non-linear equations. These have, in general, a complex array of possible dynamical patterns. In this case all parameters (fields, masses, mixing angles, correlation functions) become random variables and their behavior needs to be formulated in terms of probability distribution functions. The ability to formulate the correct noise model is critically important. For convenience, we have limited the discussion to the generic case of Levy noise.
(4) Finally, the presence of long-range interactions in space and time (extended spatial coupling, time-memory, delayed interactions) yields a problem with coupled multiple time-scales. The proper way to deal with this setting is to use the tools offered by fractional calculus and fractional dynamics [16–23] or, equivalently, with the formalism of non-extensive statistical physics [13].

Appendix A

Within the framework of perturbative QCD, the flow of the effective coupling strength with the sliding energy scale μ is governed by the beta-function:

$$\mu\frac{\partial\alpha_s}{\partial\mu} = \beta(\alpha_s) \tag{A1}$$

where

$$\begin{gathered}\beta(\alpha_s) = -b_0\alpha_s^2 - b_1\alpha_s^3 - b_2\alpha_s^4 + \mathrm{O}(\alpha_s^5)\\ b_0 = \frac{11 - \frac{2}{3}n}{2\pi},\quad b_1 = \frac{51 - \frac{19}{3}n}{4\pi^2},\quad b_2 = \frac{2857 - \frac{5033}{9}n + \frac{325}{27}n^2}{64\pi^3}\end{gathered} \tag{A2}$$

and n is the effective number of quark flavors [8]. Likewise, the scale dependence of a running quark mass $m(\mu)$ is represented by

$$\mu\frac{\mathrm{d}m(\mu)}{\mathrm{d}\mu} = -\gamma(\alpha_s)m(\mu) \tag{A3}$$

in which

$$\begin{aligned}&\gamma(\alpha_s) = c_0\alpha_s + c_1\alpha_s^2 + c_2\alpha_s^3 + \mathrm{O}\left(\alpha_s^4\right)\\ &c_0 = \frac{2}{\pi}, \quad c_1 = \frac{\frac{101}{12} - \frac{5}{18}n}{\pi^2}, \quad c_2 = \frac{1}{32\pi^3}\left[1249 - \left(\frac{2216}{27} + \frac{160}{3}\varsigma(3)\right)n - \frac{140}{81}n^2\right]\end{aligned} \tag{A4}$$

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On the relationship between Hamiltonian chaos and classical gravity

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Abstract

It is known that Hamiltonian equations of motion for low-dimensional chaotic systems are typically formulated using fractional derivatives. The evolution of such systems is governed by the fractional diffusion equation, which describes self-similar and non-Gaussian processes with strong intermittencies. We confirm, in this context, that the dynamics of a Brownian particle driven by space-time dependent fluctuations evolves towards Hamiltonian chaos and fractional diffusion. The corresponding motion of the particle has a time-dependent and nowhere vanishing acceleration. Invoking the equivalence principle of general relativity leads to the conclusion that fractional diffusion is locally equivalent to a transient gravitational field. It is shown that gravity becomes renormalizable as Newton's constant converges towards a dimensionless quantity.

1. Introduction

Fractional diffusion equations have emerged in recent years as a powerful tool for the analysis of stochastic processes and complex dynamics. In particular, fractional diffusion has been successfully linked to the study of Hamiltonian chaos in low-dimensional systems [1–4,10,11]. In this work we investigate an unexpected connection that may be established between Hamiltonian chaos and the classical theory of gravitation. The object of study is the Brownian motion of a free non-relativistic particle evolving in an environment that is random and space-time dependent. Despite its simplicity, this model offers a convenient benchmark for probing dissipative systems of higher complexity. [1]

Our three main findings are that (i) fluctuations are capable of migrating Brownian motion into Hamiltonian chaos, (ii) the Brownian particle moves as if subjected to a locally transient gravitational field and (iii) Newton's constant converges towards a dimensionless quantity as the dynamics makes the transition from fractional to the classical regime. The last finding opens the door for full renormalization of the theory, in manifest contrast with quantum gravity. The approach may be extended to include open dynamical systems and stochastic field models and may thus provide valuable insights into the long-standing issue of unification in field theory [25–28]. This is particularly attractive in light of the recently discovered decoherence mechanism responsible for the transition from quantum to classical behavior in systems strongly coupled to their environment [6,7].

It is instructive to point out that our conclusions are consistent with El Naschie's conjecture on the connection between gravitation and the Cantorian topology of space–time on or above the Planck scale ($M_{\mathrm{Pl}} \sim 10^{19}$ GeV)

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[1] We recall that, in general, there is a large spectrum of persistent fluctuations that may perturb the evolution of any dynamical system in a variety of physical settings. Examples include thermal fluctuations in statistical ensembles, Poincare resonances [21] and vacuum fluctuations in quantum physics [8].

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[12,22,24]. Our results are also relevant for theories concerned with the statistical nature of gravitational interaction in ultra-high energy physics. These models are based upon the prediction that the underlying structure of space–time undergoes large stochastic fluctuations as a result of short-distance gravitational effects [29,30].

The outline of the paper is as follows: Section 2 derives the relationship between the Langevin equation of Brownian motion and Hamiltonian chaos. A brief review of the classical Hamilton–Jacobi formalism is outlined in Section 3. The generalization of Hamilton–Jacobi equation to fractional diffusion is presented in Section 4. Section 5 establishes the explicit connection between Hamiltonian chaos and classical gravity. Renormalization is discussed in Section 6 and concluding remarks are presented in Section 7.

2. Noise driven dynamics and Hamiltonian chaos

It is well known that classical Langevin equation describes the transport of a non-relativistic Brownian particle moving in a dissipative and disordered environment [13]. Let m_0 denote the mass of the particle, γ the damping coefficient and $\eta(x,t)$ the stochastic force exerted on the particle. If there are no external potentials and the motion takes place in one dimension, the Langevin equation reads

$$m_0\ddot{x} + \gamma\dot{x} = \eta(x,t) \tag{1}$$

It is customary to assume that the stochastic force has a noise-like distribution characterized by a constant average and a shift-invariant correlation function

$$\begin{aligned} &\langle \eta(x,t)\rangle = \text{const.} \\ &\langle \eta(x,t)\eta(x',t')\rangle = Dw_x(x-x')w_t(t-t') \end{aligned} \tag{2}$$

The fluctuation–dissipation theorem requires [13,14]

$$D \sim \gamma kT \tag{3}$$

where D are the diffusion coefficient and T the temperature. [2]

A convenient noise representation is provided by the delta-kicked model [15]. Under the most general circumstances, the function $\eta(x,t)$ may be factored as

$$\eta(x,t) = \zeta(x)\sum_{n=0}^{\infty}\delta(t-n\tau) \tag{4}$$

in which $\tau = 2\pi/\Omega$ stands for the period separating successive kicks and the space-dependent amplitude is considered a superposition of power terms

$$\zeta(x) = \sum_{m=0}^{\infty} a_m x^m \tag{5}$$

The sum of delta-kicks may be expanded in harmonics of Ω to obtain

$$\sum_{n=0}^{\infty}\delta(t-n\tau) = 1 + 2\sum_{n=1}^{\infty}\cos(n\Omega t) \tag{6}$$

In what follows we assume that, on a suitably chosen observation scale, the fundamental noise mode ($n=1$) is predominant and the rest of harmonics cancel out by destructive interference. As a result, the following condition holds

$$\sum_{n=2}^{\infty}\sum_{m=0}^{\infty} a_m x^m \cos(n\Omega t) \to 0 \tag{7}$$

[2] It is important to emphasize that, according to the fluctuation–dissipation theorem, any system undergoing random perturbations must include damping as a mechanism for relaxation towards thermal equilibrium.

The Langevin model may thus be transformed into a set of coupled differential equations using the parameterization

$$\begin{aligned} y_1 &= \dot{x} \\ y_2 &= \dot{y}_1 = \frac{1 + 2\cos(y_3 t)}{m_0}\left(\sum_{m=0}^{\infty} a_m x^m\right) - \frac{\gamma}{m_0} y_1 \\ y_3 &= \Omega \end{aligned} \tag{8}$$

The system (8) resembles the evolution equations for the damped driven pendulum [16]. It has a three-dimensional phase space which is the minimum dimension required for the onset of chaos in solutions of differential equations. According to the KAM theory, the winding number

$$w = \sqrt{\frac{\gamma}{m_0 \Omega^2}} \tag{9}$$

controls the transition from unperturbed motion to weak and fully developed Hamiltonian chaos [16,17]. A manifest example of such a transition is driving with a time-dependent noise frequency $\Omega(t)$. The corresponding phase space has a rich topological structure characterized by a mixture of periodic orbits layered between chaotic islands. Fluctuations in the driving frequency generated over short time intervals lead to progressive instability and eventual breakup of KAM tori [17,23]. It is of interest to mention that the last torus destroyed by noise corresponds to the most irrational winding number, i.e. to the golden mean

$$\phi = \frac{\sqrt{5} - 1}{2} \tag{10}$$

which is a key concept of the E^{∞} theory (see [22] and included references).

3. Overview of the classical Hamilton–Jacobi formalism

It is instructive, at this point, to bridge the gap between the Langevin formalism previously outlined and the canonical approach of classical mechanics based on the Hamilton–Jacobi equation. Consider the previous example of a free non-relativistic particle of mass m_0 moving in one dimension from origin to (x, t). In the absence of any damping and disorder, its trajectory is given by

$$x(t) = \dot{x}t \tag{11}$$

The action $S(x, t)$ satisfies the Hamilton–Jacobi equation [5]

$$\frac{\partial S}{\partial t} + \frac{1}{2m_0}\left(\frac{\partial S}{\partial x}\right)^2 = 0 \tag{12}$$

and has the explicit form

$$S(x, t) = \frac{m_0}{2}\frac{x^2}{t} + S_0 \tag{13}$$

where S_0 is an arbitrary additive constant. Setting

$$\begin{aligned} -\frac{\partial S}{\partial t} &= \frac{p^2}{2m_0} = \text{const.} \\ p &= m_0 \dot{x} \end{aligned} \tag{14}$$

recovers the uniform motion expressed by (11).

As it is known, the Hamilton–Jacobi equation may be converted to a second-order partial differential equation describing standard diffusion or wave propagation. To elaborate on this point we proceed by analogy with the path integral formalism of quantum mechanics [8,31,32]. The probability amplitude for a given space–time path $x(t)$ is given by

$$\rho[x(t)] = \rho_0 \exp\{\mathrm{i}S[x(t)]\} \tag{15}$$

Assuming that the technique of analytic continuation is applicable [9,32], (15) becomes

$$\rho[x(t)] = \rho_0 \exp\{-S_{\mathrm{E}}[x(t)]\} \tag{16}$$

where $S_E(\bullet)$ represents the Euclidean action. Taking into account that momentum is a constant of motion, or $\partial^2 S_E/\partial x^2 \to 0$, the Hamilton–Jacobi equation (12) assumes the form

$$\frac{\partial\rho}{\partial t} + \frac{1}{2m_0}\frac{\partial^2\rho}{\partial x^2} = 0 \tag{17}$$

For sufficiently small space–time paths the probability amplitude is proportional to the action, that is

$$\rho[\Delta x(t)] \approx \rho_0\{1 - S_E[\Delta x(t)]\} \tag{18}$$

We shall use relation (18) in the next section.

4. Generalization of Hamilton–Jacobi formalism to fractional diffusion

For the sake of clarity we briefly summarize results obtained so far. It was found in Section 2 that, if conditions required by KAM theory are met, path dependent fluctuations are capable of migrating the classical Brownian motion into Hamiltonian chaos. The adequate formulation of this noise-driven regime requires use of fractional space and time derivatives. Section 3 has pointed out that the canonical treatment of motion in classical mechanics is based upon the Hamilton–Jacobi equation. A natural question arises on how to properly apply the Hamilton–Jacobi formulation to Hamiltonian chaos. This is the object of the current section.

Let $P(x,t)$ represent the probability density function of finding the particle at location x at instant t. Fractional diffusion equation is defined by two critical exponents (α,β) corresponding to the space and time derivatives of $P(x,t)$ [1,2]. To simplify the presentation and without any loss of generality, we set below $m_0 = \frac{1}{2}$ in (17). Fractional diffusion of the Brownian particle then takes the form

$$\frac{\partial^\beta P}{\partial t^\beta} = \frac{\partial^\alpha P}{\partial|x|^\alpha} + \frac{t^{-\beta}}{\Gamma(1-\beta)}\delta(x) \tag{19}$$

for positive time intervals $t > 0$ and point-like source functions [1,2,4]. Particular cases include Levy transport ($\beta = 1$) and fractal Brownian motion ($0 < \beta < 1$, $\alpha = 2$). The probability density stays positive if the range of the two exponents is limited to the intervals below

$$\begin{aligned} &0 < \alpha \leqslant 2 \\ &0 < \beta \leqslant 1 \end{aligned} \tag{20}$$

To simplify the formalism we adopt below the hypothesis that the integral over all possible paths connecting the initial and final space-time points can be approximated by a single contribution arising from the most dominant path. Let Δ represent the linear extent of the particle motion. Following Section 3, we note that $P(x,t)$ is equivalent to

$$P(x,t) = \frac{1}{\Delta}\left(\frac{\rho[x(t)]}{\rho_0}\right)^2 \tag{21}$$

and satisfies the normalization condition

$$\int_{-\infty}^{\infty} P(x,t)\,\mathrm{d}x = 1 \tag{22}$$

For sufficiently small paths we have from (18)

$$P(\Delta x,t) \approx \frac{1}{\Delta}\{1 - 2S_E[\Delta x(t)]\} \tag{23}$$

which shows that, up to an additive constant and a scaling factor, the probability density function and Euclidean action are identical. Under these circumstances, the asymptotic solution of the fractional diffusion equation (19) reads [2]

$$S_E[\Delta x(t)] \approx \left(-\frac{\Delta}{2}\right)\left[\frac{1}{\pi}\frac{t^\beta}{|\Delta x|^{\alpha+1}}\frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)}\sin\frac{\pi\alpha}{2}\right] \tag{24}$$

for ultra-short time intervals obeying

$$(\Delta x)^{\alpha} \gg t^{\beta} \tag{25}$$

Dimensional analysis of (24) in light of normalization (22) leads to

$$[t]^{\beta} = [\Delta x]^{\alpha} \tag{26}$$

in which [•] stands for the unit of time and space.

5. Fractional diffusion as locally transient non-inertial motion

We may naturally associate the following Hamilton–Jacobi equation to the Euclidean action (24) [18]:

$$E_{\mathrm{fr}} = -\frac{\partial^{\beta} S_{\mathrm{E}}}{\partial t^{\beta}} = \frac{m_0}{2}\left[\frac{\partial^{\beta}(\Delta x)}{\partial t^{\beta}}\right]^2 \tag{27}$$

where E_{fr} is the energy transported by the fractional diffusion process and $\frac{\partial^{\beta}(\Delta x)}{\partial t^{\beta}}$ generalizes the ordinary velocity corresponding to $\beta = 1$. Hence

$$v_{\mathrm{fr}} = \frac{\partial^{\beta}(\Delta x)}{\partial t^{\beta}} = \sqrt{2}\left[\Delta \frac{\Gamma(1+\alpha)\sin(\pi\alpha/2)}{\pi|\Delta x|^{1+\alpha}}\right]^{1/2} \tag{28}$$

Following the rules of fractional differentiation [20], the generalized acceleration may be obtained from (28) as

$$a_{\mathrm{fr}} = \frac{\partial^{\beta} v_{\mathrm{fr}}}{\partial t^{\beta}} = v_{\mathrm{fr}} \frac{t^{-\beta}}{\Gamma(1-\beta)} \tag{29}$$

This expression indicates that the free Brownian particle undergoes a space-time dependent non-inertial motion for $t < \infty$. The fractional acceleration vanishes in the limit $\beta = 1$ as $\Gamma(0) \to \infty$. According to the equivalence principle of general relativity, a non-inertial frame of reference is locally identical to a gravitational field. We conclude that, under the assumption that the equivalence principle holds for non-smooth trajectories, the statistical transport of the free Brownian particle is locally equivalent to the action of a transient gravitational field. The next section attempts to show that this field may be described by a renormalizable theory.

6. Dimensional analysis and renormalization

In the relativistic theory of gravitation Newton's constant carries a negative mass dimension. Power expanding the metric around the Lorentz solution leads to a non-polynomial action in this constant [19,32] (see Appendix A). As a result, quantum gravity theories founded on general relativity are considered non-renormalizable. The object of this section is to evaluate the impact of critical exponents (α, β) on renormalizability from arguments based on dimensional analysis.

(26)–(28) may be used to determine the dimensions of energy, fractional velocity and mass starting from the scalar nature of the Euclidean action. We find, respectively

$$\begin{aligned}
[E_{\mathrm{fr}}] &= t^{-\beta} \\
[v_{\mathrm{fr}}] &= [\Delta x][t]^{-\beta} = [\Delta x]^{1-\alpha} \\
[m_0] &= \frac{[E_{fr}]}{[v_{\mathrm{fr}}]^2} = [\Delta x]^{\alpha-2}
\end{aligned} \tag{30}$$

In order to include Newton's constant in these considerations it is necessary to write down a fundamental field equation. The most straightforward choice is the Poisson equation of classical field theory. Let Φ_{fr} and G_{fr} represent the gravitational potential and coupling constant induced by fractional diffusion. The natural generalization of Poisson's equation in $1+1$ space–time is

$$\frac{\partial^{2\alpha}\Phi_{\mathrm{fr}}}{\partial(\Delta x)^{2\alpha}} = 4\pi G_{\mathrm{fr}}\rho \tag{31}$$

where ρ is the equivalent source of Φ_{fr}, expressed in units of mass per unit of length. [3] Since the standard Poisson equation is recovered in the limit $\alpha = 1$, it makes sense to change the upper bound in (20) such that $0 < \alpha \leqslant 1$. The solution of (31) for a uniform source and subject to the boundary condition

$$\Phi_{\mathrm{fr}}(\Delta x = 0) = \Phi_0 \tag{32}$$

is supplied by [20]

$$\Phi_{\mathrm{fr}}(\Delta x) = 4\pi G_{\mathrm{fr}}\rho \frac{(\Delta x)^{2\alpha}}{\Gamma(2\alpha + 1)} + \Phi_0 \tag{33}$$

In general relativity the gravitational potential is dimensionless ([5] and Appendix A) which can be expressed as

$$[\Phi] = [m_0]^0 \tag{34}$$

In the framework provided by fractional diffusion this constraint may be relaxed to a less restrictive requirement, that is

$$[\Phi_{\mathrm{fr}}] = [m_0]^{\gamma(\alpha)} \tag{35}$$

where $\gamma(\alpha)$ represents an α-dependent exponent obeying

$$\lim_{\alpha \to 1} \gamma(\alpha) = 0 \tag{36}$$

It is apparent that condition (36) does not uniquely determine the explicit form of $\gamma(\alpha)$. For example, two choices from the infinite span of possible solutions are

$$\begin{aligned} \gamma(\alpha) &= 1 - \alpha^2 \\ \gamma(\alpha) &= |\ln \alpha| \end{aligned} \tag{37}$$

As it is shown below, we use this redundancy to control the mass dimension of G_{fr}.
Since

$$[\rho] = \frac{[m_0]}{[\Delta x]} \tag{38}$$

we obtain from (30), (33), (35) and (38)

$$[G_{\mathrm{fr}}] = [m_0]^{\gamma(\alpha) - \frac{3(\alpha-1)}{\alpha-2}} \tag{39}$$

Demanding a positive or vanishing mass dimension in (39) amounts to

$$\gamma(\alpha) \geqslant \frac{3(\alpha - 1)}{\alpha - 2} \tag{40}$$

which further restricts the space of acceptable functions $\gamma(\alpha)$.

Using (36) it is seen that condition (40) is automatically satisfied for $\alpha \to 1$, that is, when the dynamics makes the transition from fractional to the classical regime.

It is instructive to consider the particular choice $\gamma(\alpha) = |\ln \alpha|$. Condition (40) leads to

$$\alpha \leqslant 0.28683 \tag{41}$$

7. Concluding remarks

We have reported the close connection between Hamiltonian chaos and fractional diffusion, on the one hand, and classical theory of gravitation on the other. It was found that fractional diffusion enables Newton's constant to converge towards a dimensionless quantity and creates the necessary framework for renormalization. The approach is built upon the Hamilton–Jacobi formalism and may be thus extrapolated to a larger class of field theories. Our work complements

[3] A similar analysis may be carried out in $3 + 1$ space–time. It involves a lengthy derivation and it is not included here.

similar studies linking classical gravity to space–time fluctuations, as well as several papers on unification via fractal topology [24–28].

Appendix A

For ease of reading we briefly review in this Appendix A some key points regarding the renormalization topic of quantum gravity and related theories. Additional details may be found in [8,31,32].

The potential generated by a point mass m at a distance R in Newtonian gravitation is given by

$$\varphi = -G\frac{m}{R} \tag{A.1}$$

Let $g_{\mu\nu}$ denote the components of the metric tensor ($\mu, \nu = 0, 1, 2, 3$). The potential is a dimensionless quantity related to the magnitude of g_{00}, the temporal component of the metric tensor, via

$$g_{00} = 1 + 2\varphi \tag{A.2}$$

Because g_{00} and φ are both dimensionless and since, in natural units, distance is measured as reciprocal of mass

$$[R] = [m]^{-1} \tag{A.3}$$

it follows from (A.1) that Newton's constant has a -2 mass dimension, that is

$$[G] = [m]^{-2} \tag{A.4}$$

The negative mass dimension carried by G makes gravity non-renormalizable due to the following argument: the probability amplitude for graviton-graviton scattering at a given energy E may be computed using the series expansion

$$\text{amplitude} \sim 1 + GE^2 + (GE^2)^2 + \cdots \tag{A.5}$$

where different orders correspond to various Feynman diagrams. The series (A.5) is manifestly divergent and the resulting scattering amplitude lacks physical meaning.

A similar argument may be brought up in conjunction with any attempt to quantize gravity by power expanding the metric tensor $g_{\mu\nu}$ around the Euclidean metric $g^{(0)}_{\mu\nu}$ (where $g^{(0)}_{\mu\nu}$ is referred to as the Lorentz solution)

$$g_{\mu\nu}(x) = g^{(0)}_{\mu\nu} + \sqrt{G}h_{\mu\nu} \tag{A.6}$$

In the above, the metric deviations $h_{\mu\nu}$ are associated with the graviton field. Each term of the series contains derivatives and an ever-increasing number of $h_{\mu\nu}$ fields and powers of G. The action series assumes the generic form

$$S \sim \frac{1}{16\pi G}\int d^4x[\partial h\partial h + h\partial h\partial h + h^2\partial h\partial h + \cdots] \tag{A.7}$$

where space–time indexes μ, ν have been omitted for simplicity. The action expansion is not considered polynomial due to the very existence of a non-scalar Newton constant.

Dimensional analysis indicates that a renormalizable theory must be characterized by a coupling constant having a positive or vanishing mass dimension. Quantum electrodynamics, the electroweak model and quantum chromodynamics are examples of renormalizable theories because the fine-structure constant and gauge couplings g_1, g_2 and g_3 are dimensionless.

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Renormalization group and the emergence of random fractal topology in quantum field theory

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Abstract

This work reveals the close connection between the random fractal topology of space–time in microphysics and the renormalization group program (RG) of quantum field theory. As known, the primary goal of RG is to consistently remove divergences from quantum computations by factoring in the energy scale (μ) at which physical processes are probed. RG postulates that the action functional is independent of any particular choice of μ, that is, physical processes are invariant to arbitrary changes of the observation scale. In this context, we conjecture that μ represents a continuous random variable having a uniform density function. Novel results emerge in the basin of attraction of all fixed points, namely: (i) the field exponent becomes a continuous random variable and (ii) space–time coordinates become fractals with random dimensions. It is concluded that the random topology of space–time is not an exclusive attribute of the Planck scale but an inherent manifestation *of stochastic dynamics* near any fixed point of the underlying field theory.

1. Introduction

In recent years significant effort has been devoted to applications of fractal geometry, deterministic chaos and stochastic dynamics in classical and quantum physics. Due to the wide extent of this research field, a complete listing of main contributions is impractical. We mention here few examples that are representative for the topic of this work: anomalous diffusion and Levy statistics in Hamiltonian phase space [12,13], non-differentiability of Feynman paths [14,15], application of fractional Brownian motion in quantum field theory [4], vacuum fluctuations, chaotic maps and stochastic quantization in particle physics [16], mass generation in the lepton sector due to period doubling transition to chaos [17], quantum Brownian motion [18], fractional dynamics and origin of the fine-structure constant [19], Cantorian space–time and the topological foundation of coupling and mass spectra in the Standard Model [20–27].

The prevailing interpretation of El Naschie's E^{∞} model is that the Cantorian space–time topology emerges at mass scales comparable to the Planck length. Drawing on recent results regarding renormalization group in the presence of quantum fluctuations [4], our work suggests that the random topology of space–time is not an exclusive attribute of the Planck scale but an inherent manifestation *of stochastic dynamics* near any fixed point of the underlying field theory.

The paper is organized as follows: Section 2 introduces the concept of random observation scale from arguments related to equilibrium statistical mechanics. Taking the φ^4 theory as a benchmark model, Section 3 examines the behavior of the RG solution near the unique fixed point $g \to 0$. The stochastic character of the field exponent is analyzed in Section 4. Section 5 investigates the temporal evolution of the φ field from a statistical mechanics perspective. Connection of space–time coordinates to fractal objects having random dimension is discussed in Section 6. Results are summarized in Section 7.

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2. The random observation scale: a statistical mechanics argument

Classical statistical mechanics of systems at thermal equilibrium asserts that energy or energy and number of particles in thermodynamic ensembles are subject to incessant fluctuations. Consider, for example, a system enclosed in a heatbath at constant finite temperature T (a "canonical ensemble"). The energy fluctuation is linearly dependent on temperature according to [1]:

$$\frac{\Delta E}{\langle E\rangle} \sim \frac{T}{N\varepsilon} \tag{1}$$

where $\langle E\rangle$ is the thermal average of the energy, N is the number of particles in the system and ε is the energy per particle. Fluctuations vanish for macroscopic systems in the limit $N \to \infty$ where the distribution of energy is sharply peaked around $\langle E\rangle$. In contrast, fluctuations survive in canonical ensembles comprising low-dimensional classical or quantum systems such as dilute Bose gases.

In the same context, we recall that quantum fields are objects with manifest statistical properties owing to the cascade of virtual processes that occur during propagation and interaction. We also recall that a basic requirement of any realistic quantum field theory is renormalizability [2]. To render all computations finite, a well-established regularization procedure needs to be implemented. The renormalization program imposes an arbitrary energy scale (μ) upon which no physical consequences must depend [2,3]. This sets the "coarse-graining" scale and the resolution at which the underlying physics is probed.

From these arguments it follows that, if the average energy of the quantum field system $\langle E\rangle$ sets its temperature, then μ is expected to undergo continuous fluctuations about $\langle E\rangle$. There are no preferential values in this random occurrence. Hence, invoking the uncertainty principle, we assert below that μ represents a continuous random variable with a uniform probability density function $p(\mu)$. Let μ be defined inside a range bounded by μ_M and μ_m. We have

$$\begin{aligned} &p(\mu) = \frac{1}{\mu_M - \mu_m} \leqslant \frac{\tau}{h} \quad \text{if } \mu \in \{\mu_m, \mu_M\} \\ &p(\mu) = 0 \quad \text{if otherwise} \end{aligned} \tag{2}$$

in which h is Planck's constant and τ sets the observation time window. During τ we assume that the system randomly samples all available energy scales contained in the range.

A comment on (2) is now in order. Dimensional consistency requires the right hand side of (2) to be expressed in scalar form. The simplest way to fulfill this requirement is to work with a relative energy scale μ^0 defined via

$$\mu^0 = \frac{\mu}{\mu_M} \tag{3}$$

which converts (2) into

$$\begin{aligned} &p(\mu^0) = \frac{1}{1 - \mu_m^0} \leqslant \tau^0 \quad \text{if } \mu \in \{\mu_m, \mu_M\} \\ &p(\mu^0) = 0 \quad \text{if otherwise} \end{aligned} \tag{4}$$

where $h = 1$ and $\tau^0 = \mu_M\tau$ is the non-dimensional time observation window. According to this interpretation, the time window τ^0 and energy scale μ^0 are statistically conjugate variables. This implies that there is random sampling of all available time instants contained in τ^0 in a similar fashion with energy sampling in the range $[\mu_m^0, 1]$. The natural outcome of this conjecture is that time behaves as a stochastic variable. To derive similar conclusions on the space coordinate we note that, in non-relativistic field theories, time and space scale independently as

$$\begin{aligned} &\vec{x} = s\overrightarrow{x'} \\ &t = s^z t' \end{aligned} \tag{5}$$

where $z \neq 1$ is the so-called dynamic exponent [4]. In scalar form (5) reads

$$\begin{aligned} &\vec{x}^0 = s\overrightarrow{x'}^0 \\ &t^0 = s^z t'^0 \end{aligned} \tag{6}$$

in which

$$x^0 = x\mu_M \qquad t^0 = t\mu_M \tag{7}$$

It follows that space and time are non-trivially related through

$$t^0 \sim (x^0)^z \tag{8}$$

and the analogue of (4) is given by

$$p(\mu^0) = \frac{1}{1-\mu_m^0} \leqslant (\lambda^0)^z \quad \text{if } \mu \in \{\mu_m, \mu_M\} \qquad p(\mu^0) = 0 \quad \text{if otherwise} \tag{9}$$

where $\lambda^0 = \lambda\mu_M$ is the spatial observation window. The space coordinate randomly samples all available locations contained in λ^0 as the energy scale fluctuates within $[\mu_m^0, 1]$.

3. Behavior of the RG solution near fixed points

Consider a simple field theoretic framework comprising the scalar field operator $\varphi(\vec{x}, t)$, the mass parameter m and coupling constant g. A typical prototype is the well-known φ^4 model whose free-form action is [2,3]

$$S[\varphi] = \int \left\{ \frac{1}{2}[\partial_v\varphi\partial^v\varphi - m^2\varphi^2] - \frac{\mu^{4-d}g}{4!}\varphi^4 \right\} \mathrm{d}^{d-1}x\,\mathrm{d}t \tag{10}$$

in which d stands for the dimension of the space–time domain and v is the space–time index. Fluctuations in the observation scale are expected to create subsequent fluctuations of the operator $\varphi(\vec{x}, t)$. The field probability density function $p_\mu(\varphi, g, m)$ has dimension

$$[\varphi]^{-1} = \mu_M^{1-\frac{d}{2}} \tag{11}$$

and satisfies the RG equation [4]:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \delta(g)m\frac{\partial}{\partial m} - \frac{\gamma(g)}{2}\varphi\frac{\partial}{\partial\varphi}\right] p_\mu(\varphi, g, m) = 0 \tag{12}$$

where the coefficient functions

$$\mu\frac{\partial g}{\partial\mu} = \beta(g) \qquad \mu\frac{\partial m}{\partial\mu} = \delta(g)m \qquad \mu\frac{\partial\varphi}{\partial\mu} = -\frac{\gamma(g)}{2}\varphi \tag{13}$$

outline the scale dependence of coupling, mass and field, respectively. As known, coefficient functions are specific for each field theory. In particular, the sign of $\beta(g)$ determines the type of asymptotic behavior at large momenta, that is, whether or not the theory displays asymptotic freedom [4,5]. In general, the zeros of $\beta(g)$ define the set of fixed points $\{g_i^*\},\ i = 1, 2, \ldots, N$. In the neighborhood of these points (i.e., in their basin of attraction) Eqs. (13) become

$$\mu\frac{\partial m}{\partial\mu} = \delta(g_i^*)m \qquad \mu\frac{\partial\varphi}{\partial\mu} = -\frac{\gamma(g_i^*)}{2}\varphi \tag{14}$$

To further carry out computations involving exclusively scalar variables, it is convenient to cast (14) in a non-dimensional form. Dividing both sides by their respective units for mass and field yields

$$\mu^0 \frac{\partial m^0}{\partial \mu^0} = \delta(g_i^*) m^0$$
$$\mu^0 \frac{\partial \varphi^0}{\partial \mu^0} = -\frac{\gamma(g_i^*)}{2} \varphi^0 \tag{15}$$

where

$$\varphi^0 = \frac{\varphi}{[\varphi]}$$
$$m^0 = \frac{m}{\mu_M} \tag{16}$$

(15) is solved by the following set of closed-form solutions:

$$m_{\mathrm{i}}^0 = K_m (\mu^0)^{\delta(g_i^*)}$$
$$\varphi_{\mathrm{i}}^0 = K_\varphi (\mu^0)^{-\frac{\gamma(g_i^*)}{2}} \tag{17}$$

in which K_m, K_φ are integration constants. It can be seen that, in general, both field and mass solutions display a N-fold multiplicity and acquire properties of continuous random variables. There is only one fixed point in the φ^4 model corresponding to the IR limit $(g \to 0)$ as $\mu \to 0$. In its basin of attraction we have

$$m^0 = f_m(\mu^0) = K_m (\mu^0)^{\delta_0}$$
$$\varphi^0 = f_\varphi(\mu^0) = K_\varphi (\mu^0)^{-\frac{\gamma_0}{2}} \tag{18}$$

where $\gamma_0 = \gamma(0)$ and $\delta_0 = \delta(0)$.

Since the power-law relations contained in (18) are strictly monotonic, their respective probability density functions are given by [6]

$$p(m^0) = p_\mu [f_m^{-1}(m^0)] \left| \frac{\mathrm{d}}{\mathrm{d}m^0} [f_m^{-1}(m^0)] \right|$$
$$p(\varphi^0) = p_\mu [f_\varphi^{-1}(\varphi^0)] \left| \frac{\mathrm{d}}{\mathrm{d}\varphi^0} [f_\varphi^{-1}(\varphi^0)] \right| \tag{19}$$

leading to

$$p(m^0) = \frac{K_m^{-\frac{1}{\delta_0}}}{(1-\mu_m^0)\delta_0} (m^0)^{\frac{1}{\delta_0}-1}$$
$$p(\varphi^0) = \frac{2K_\varphi^{\frac{2}{\gamma_0}}}{(1-\mu_m^0)\gamma_0} (\varphi^0)^{-\left(\frac{2}{\gamma_0}+1\right)} \tag{20}$$

One may verify from (20) that the completeness relation

$$\int_{\mu_m^0}^1 p(m^0)\,\mathrm{d}m^0 = \int_{\mu_m^0}^1 p(\varphi^0)\,\mathrm{d}\varphi^0 = 1 \tag{21}$$

is automatically satisfied. For simplicity we choose $K_m = K_\varphi = 1$.

Let $m_{\mathrm{i}}^0, \varphi_{\mathrm{i}}^0$ and $m_{\mathrm{f}}^0, \varphi_{\mathrm{f}}^0$ designate two random sets of initial and final mass and field states. Using (20) yields the following mass and field moments corresponding to these random intervals

$$\langle (m^0)^2 \rangle_{\mathrm{fi}} = \frac{1}{(1-\mu_m^0)(2\delta_0+1)} \left[(\mu_{\mathrm{f}}^0)^{2\delta_0+1} - (\mu_{\mathrm{i}}^0)^{2\delta_0+1} \right]$$
$$\langle (\varphi^0)^2 \rangle_{\mathrm{fi}} = \frac{1}{(1-\mu_m^0)(\gamma_0-1)} \left[(\mu_{\mathrm{f}}^0)^{1-\gamma_0} - (\mu_{\mathrm{i}}^0)^{1-\gamma_0} \right] \tag{22}$$

Let the initial state be fixed and the final state randomly vary. Up to an additive constant (22) may be written as

$$\begin{aligned} \langle (m^0)^2 \rangle &= \frac{1}{(1-\mu_m^0)(2\delta_0+1)} (\mu^0)^{2\delta_0+1} \\ \langle (\varphi^0)^2 \rangle &= \frac{1}{(1-\mu_m^0)(\gamma_0-1)} (\mu^0)^{1-\gamma_0} \end{aligned} \tag{23}$$

We shall make use of these relations in Section 6.

4. Stochastic nature of field scaling

A well-known property of critical phenomena is that, near transition points, all relevant variables scale in a similar fashion with the control parameter. The scaling behavior is defined by a set of fixed exponents that are deterministic in nature and dependent on the dimensionality of the system [5,7,8]. In contrast, we are going to show in this section that the field exponent γ_0 acquires a stochastic character due to the postulated randomness of the observation scale μ^0.

Let

$$G(\vec{x}^0, t^0) = \langle \varphi^0(\vec{x}^0, t^0) \varphi^0(0,0) \rangle \tag{24}$$

denote the field propagator from the initial space–time point (0,0) to the arbitrary point $(\vec{x}, t)$ [5]. Upon independent scaling of the non-dimensional space–time coordinates according to (6), it can be shown that the field propagator changes as [4]

$$G(\vec{x}^0, t^0) = (r^0)^{2\chi} f\left[\frac{t^0}{(r^0)^z}\right] \tag{25}$$

if the field scales as

$$\varphi^0(\vec{x}^0, t^0) = s^\chi \varphi^0\left(\vec{x'}^0 t'^0\right) \tag{26}$$

and the scale factor is chosen to be proportional to the modulus of the position vector

$$s \sim |\vec{x}^0| = r^0 \tag{27}$$

The asymptotic behavior of $f\left[\frac{t^0}{(r^0)^z}\right]$ is supplied by

$$\begin{aligned} &\lim f\left[\frac{t^0}{(r^0)^z}\right] \to \text{const.} \quad \text{if } t^0 \ll (r^0)^z \\ &\lim f\left[\frac{t^0}{(r^0)^z}\right] \to \left[\frac{t^0}{(r^0)^z}\right]^{2\chi/z} \quad \text{if } t^0 \gg (r^0)^z \end{aligned} \tag{28}$$

In d-dimensional space–time and at the fixed point, χ is related to the field exponent γ_0 via

$$\chi = -\left(\frac{d}{2} - 1 + \frac{\gamma_0}{2}\right) \tag{29}$$

Using the propagator interpretation as probability density amplitude for transitions involving the initial (0,0) and final space–time location $(\vec{x}, t)$ [3], we demand

$$\int_{x_i^0}^{x_f^0} [G(\vec{x}^0, t^0)]^2 \, \mathrm{d}^{d-1} x^0 = 1 \tag{30}$$

in which (x_i^0, x_f^0) are random limits of the spatial domain. In order to explicitly carry out the integral, these limits need to be expressed in terms of μ^0. Let us rewrite (9) in the equivalent minimal form

$$[x^0 - \langle x^0 \rangle]^z [\mu^0 - \langle \mu^0 \rangle] = 1 \tag{31}$$

and take for simplicity

$$\langle x^0 \rangle = \langle \mu^0 \rangle = 0 \tag{32}$$

Thus

$$x_{\mathrm{i}}^0 = (\mu_{\mathrm{i}}^0)^{-\frac{1}{z}}$$
$$x_{\mathrm{f}}^0 = (\mu_{\mathrm{f}}^0)^{-\frac{1}{z}} \tag{33}$$

We note that (30) is a result of the general closure condition

$$\int G(\vec{x},t)G^*\left(\vec{x'},t\right)\mathrm{d}^{d-1}x = \delta^{(d-1)}\left(\vec{x}-\vec{x'}\right) \tag{34}$$

upon a suitable normalization of the δ-function to unity [9]. Since, apart from a multiplicative constant,

$$\mathrm{d}^{d-1}x^0 = (r^0)^{d-1}\mathrm{d}r^0 \tag{35}$$

we find from (30)

$$(\mu_{\mathrm{f}}^0)^{\frac{d-2(2-\gamma_0)}{z}} - (\mu_{\mathrm{i}}^0)^{\frac{d-2(2-\gamma_0)}{z}} = 2(2-\gamma_0)-d \tag{36}$$

Keeping the initial state fixed and letting the final state fluctuate, gives the generic relation

$$(\mu^0)^{\frac{d-2(2-\gamma_0)}{z}} = 2(2-\gamma_0)-d \tag{37}$$

highlighting the random nature of γ_0.

5. Statistical mechanics of the φ^4 model

In the previous section it was shown that the field exponent γ_0 is no longer a fixed parameter of the RG but a random variable dependent on μ^0. To gain further insight into statistical properties of the scalar field, it is desirable to analyze the effect of the stochastic observation scale on the field dynamics. To this end, a convenient starting point is the effective action formalism of the φ^4 theory [3]. Consider the vacuum expectation value of the operator $\varphi(\vec{x},t)$ in the presence of an external source $J(\vec{x},t)$

$$\varphi_{\mathrm{c}}(\vec{x},t) = \frac{\langle 0|\varphi(\vec{x},t)|0\rangle_J}{\langle 0|0\rangle_J} \tag{38}$$

(38) represents the classical counterpart of the scalar field operator. The equation of motion for the free φ^4 theory is obtained by replacing $\varphi_{\mathrm{c}}(\vec{x},t)$ in (10)

$$\frac{\delta S[\varphi_{\mathrm{c}}]}{\delta\varphi_{\mathrm{c}}} = [\partial_\nu\partial^\nu + m^2]\varphi_{\mathrm{c}}(\vec{x},t) + \frac{\mu^{4-d}g}{3!}\varphi_{\mathrm{c}}(\vec{x},t)^3 = 0 \tag{39}$$

For simplicity, we proceed with the assumption that the field $\varphi_{\mathrm{c}}(\vec{x},t)$ is spatially uniform, $\varphi_{\mathrm{c}} = \varphi_{\mathrm{c}}(t)$. Furthermore, to ensure consistency with the formal treatment developed so far, we pass to a non-dimensional representation of (39) by using (7), (11) and (16). The resulting equation of motion reads

$$\left[\frac{\partial^2}{\partial(t^0)^2} + (m^0)^2\right]\varphi_{\mathrm{c}}^0(t^0) + \frac{g}{3!}\varphi_{\mathrm{c}}^0(t^0)^3 = 0 \tag{40}$$

In the basin of attraction of the IR fixed point, the non-linear coupling term in (40) becomes a small perturbation to the free harmonic oscillator whose equation is

$$\left[\frac{\partial^2}{\partial(t^0)^2} + (m_{\mathrm{eff}}^0)^2\right]\varphi_{\mathrm{c}}^0(t^0) = 0 \tag{41}$$

The net result of this approximation is that the original mass may be replaced by an effective mass parameter [10]

$$m_{\mathrm{eff}}^0 = m^0 + \frac{g(\Phi_{\mathrm{c}}^0)^2}{16m^0} \tag{42}$$

where Φ_{c}^0 is the amplitude for the closed-form solution of (40), i.e.

$$\varphi_{\mathrm{c}}^0 = \Phi_{\mathrm{c}}^0 sn(m_{\mathrm{eff}}^0 t^0) \tag{43}$$

and m^0 is a random variable depending on δ_0 (Section 3).

6. Emergence of space–time as a random fractal set

The stochastic nature of φ_c^0 can be fully accounted for by adopting a statistical interpretation of its dynamics. In this context, the field second moment may be computed as [11]

$$\langle(\varphi_c^0)^2\rangle \sim \coth\left(\frac{1}{2}m_{\mathrm{eff}}^0 t^0\right) \tag{44}$$

In the IR limit $g \ll m^0 < 1$ and for time intervals consistent with (28) (that is, for $t^0 \ll (r^0)^z$), a reasonable approximation of (44) is

$$\langle(\varphi_c^0)^2\rangle \sim \frac{1}{m_{\mathrm{eff}}^0 t^0} \tag{45}$$

Direct comparison of (23) with (45) gives

$$t^0 \sim (1-\mu_m^0)(\gamma_0 - 1)t_{\mathrm{eff}}^0(\mu^0)^{\gamma_0 - 1} \tag{46}$$

in which $t_{\mathrm{eff}}^0 = (m_{\mathrm{eff}}^0)^{-1}$ is the period of the harmonic oscillator described by (41). Using (8) we find

$$x^0 \sim (1-\mu_m^0)(\gamma_0 - 1)t_{\mathrm{eff}}^0(\mu^0)^{\frac{\gamma_0 - 1}{z}} \tag{47}$$

It follows from (46) and (47) that time and space coordinates scale as random fractals whose dimensions depend on both δ_0 and γ_0.

7. Summary

Starting from the viewpoint that the renormalization group scale is a continuous random variable spanning a specific range, we have shown that, near fixed points of the underlying field theory, the space–time manifold acquires properties of random fractal sets. We have found that the manifold dimension depends on the values taken by the mass and field exponents at the fixed point. The φ^4 theory has been chosen as an illustrative framework, however, results are not restricted to this model.

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Stability of renormalization group trajectories and the fermion flavor problem

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Abstract

A long-standing puzzle of the current standard model for particle physics is that both leptons and quarks arise in replicated patterns. Our work suggests that the number of fermion flavors may be directly derived from the dynamics of coupling flow equations. Specifically, we argue that the number of flavors results from demanding stability of the coupling flow about its fixed-point solution.

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Keywords: Renormalization Group; Gauge field theories; QED; QCD; Stability of the laminar flow

1. Introduction and motivation

The standard model for particle physics (SM) represents a highly successful framework for the description of sub-nuclear particles and their interactions in an energy range bounded by an upper limit of about 100 GeV ([10] and Appendix A). The backbone of SM is relativistic quantum field theory (QFT) whose predictive power rests primarily on the techniques of perturbation theory [1–7,19]. A key premise of QFT is that the cumulative contribution of arbitrary-order quantum corrections above any energy threshold can be conveniently suppressed. Carrying out this program means that all quantum processes above the threshold can be absorbed into a redefinition of parameters that make up the theory (masses, couplings, fields). It is customary to call this prescription the "renormalization group" approach (RG) and its outcome an "effective field theory". The main outcome of RG is that the parameters of the theory depend on the energy scale at which the physics is probed ([8] and Appendix B). In particular, an important concept in RG is the evolution of coupling with the energy scale, referred to as the coupling flow equation. Since SM is an effective framework for

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the description of particle physics below 100 GeV [1–7], it is typically assumed that the coupling flow is stable and its approach towards equilibrium develops in a deterministic way.

Despite its remarkable predictive power, SM cannot explain why both leptons and quarks arise in replicated patterns. This puzzle is referred to as the fermion ‘‘flavor problem’’ [11,19] and it continues to challenge to the day our understanding of particle physics. Motivated by the relevance of nonlinear dynamics in field theory [12–17,20,21], this work suggests that the number of fermion flavors may be directly derived from the dynamics of coupling flow equations. Specifically, we find that the number of flavors results from demanding stability of the coupling flow about its fixed-point solution.

The paper is organized as follows. Section 2 surveys the theory underlying the gauge coupling flow in SM and Section 3 retrieves the number of fermion flavors from a standard stability analysis. Results and concluding remarks are detailed in the last two sections. Three Appendix sections are included for convenience. The deal, respectively, with a brief overview of SM, an introduction to the RG approach to coupling flow equations and a brief presentation of the Routh–Hurwitz criterion.

2. Coupling flow equations

We start from the set of beta-functions describing the RG flow in the gauge sector of SM [1–7, Appendix B]:

$$\frac{\mathrm{d}g_i}{\mathrm{d}t} = \beta_i(g) = b_i(N,n)g_i^3 + \mathrm{O}(g_i^5) \tag{1}$$

in which $i = (1,2,3)$, N is the dimension of the gauge group and n the number of fermion flavors. In particular, the beta-functions for quantum electrodynamics (QED) and non-abelian gauge theories (the weak interaction model and QCD) are respectively supplied by [6,7]

$$\beta_{\mathrm{QED}}(e) = N\frac{ne^3}{12\pi^2} + \mathrm{O}(e^5), \tag{2}$$

$$\beta_{\mathrm{NA}}(g) = -\frac{11N-2n}{48\pi^2}g^3 + \mathrm{O}(g^5). \tag{3}$$

Accounting for the underlying $SU(3)\times SU(2)\times U(1)$ gauge structure of SM, the explicit form of the coefficient vector is

$$\mathbf{b}(N,n) = \begin{vmatrix} b_1(1,n) \\ b_2(2,n) \\ b_3(3,n) \end{vmatrix} \tag{4}$$

with entries

$$\begin{aligned} b_1(1,n) &= \frac{n}{12\pi^2}, \quad \text{for } N=1, \\ b_2(2,n) &= -\frac{11-n}{24\pi^2}, \quad \text{for } N=2, \\ b_3(3,n) &= -\frac{33-2n}{48\pi^2}, \quad \text{for } N=3. \end{aligned} \tag{5}$$

Let us assume in what follows that typical coupling strengths of SM represent fixed-point solutions of (2) and (3). For reference, we also assume that these are computed at the high-energy limit set by the mass of the Z boson [Appendix B, 19]:

$$\begin{aligned} \alpha_{\mathrm{QED}}(M_Z) &= 1/127.9 \approx 0.00782, \\ \alpha_2(M_Z) &= 0.0338, \\ \alpha_3(M_Z) &= 0.123, \end{aligned} \tag{6a}$$

or, in set form

$$\alpha(M_Z) = \{\, 0.00782 \quad 0.0338 \quad 0.123 \,\} \tag{6b}$$

Using (6b) the set of coupling parameters is given by

$$g_0^2(M_Z) = 4\pi\alpha(M_Z) = \{\,0.098 \quad 0.425 \quad 1.546\,\} \tag{7}$$

3. Stability analysis

The set of three nonlinear differential Eqs. (2) and (3) based on (5) and (7) depends on the number of flavors n, which plays the role of an independent control parameter. Qualitative changes in the behavior of coupling trajectories are to be expected when n is finely tuned. As pointed out in Section 1, a typical assumption made in QFT is that the coupling flow evolves towards a finite set of attractors consisting of isolated fixed points [17]. On this basis we require that (2) and (3) yield a coupling flow that is *unique* and *stable*. These constraints amount to demanding that all Lyapunov exponents are real and vanishing with the exception of a single one, which is either vanishing or negative. Expanding (2) and (3) about (7) yields the new coefficient vector

$$\mathbf{a}(N,n) = 3g_0^2(M_Z)\mathbf{b}(N,n) = 10^{-3}\begin{vmatrix} 2.482n \\ -5.382(11-n) \\ -9.790(33-2n) \end{vmatrix}. \tag{8}$$

Following the Routh–Hurwitz criterion, the set of stability parameters assumes the form (see Appendix C):

$$\begin{aligned} p(n) &= -[a_{11}(n) + a_{22}(n) + a_{33}(n)], \\ q(n) &= [a_{11}(n)a_{22}(n) + a_{11}(n)a_{33}(n) + a_{22}(n)a_{33}(n)], \\ r(n) &= -a_{11}(n)a_{22}(n)a_{33}(n), \end{aligned} \tag{9}$$

where $a_{kk}(n)$, $k = 1, 2, 3$ are supplied by the components of (8). The characteristic equation is represented by the cubic polynomial

$$\Delta(n) = \lambda^3 + p(n)\lambda^2 + q(n)\lambda + r(n) = 0. \tag{10}$$

The constraint of a unique and stable trajectory implies

$$\lambda_1 = \lambda_2 = 0, \tag{11a}$$

$$\lambda_3 \leqslant 0, \tag{11b}$$

which yields

$$\begin{aligned} p(n) &\geqslant 0, \\ q(n) &= r(n) = 0. \end{aligned} \tag{12}$$

We obtain the least squares solution

$$\boxed{n \cong 7.3} \tag{13}$$

4. Discussion of results

Fig. 1 graphs the variation of the stability parameters with the flavor number. As expected, the least squares solution lies at the intersection point of $q(n)$ and $r(n)$. There are two distinct interpretations of this result, namely:

(1) the actual number of flavors in SM is indeed seven and so we should anticipate an extra fermion flavor to be discovered in future accelerator experiments (such as, but not limited to, the fourth family neutrino [9]);
(2) the stability analysis we have developed is only an approximation that needs further revision. One can invoke here, for example, including higher-order corrections to (2) and (3), accounting for the Yukawa sector of the coupling flow [19] or starting from the framework of non-perturbative RG flow equations

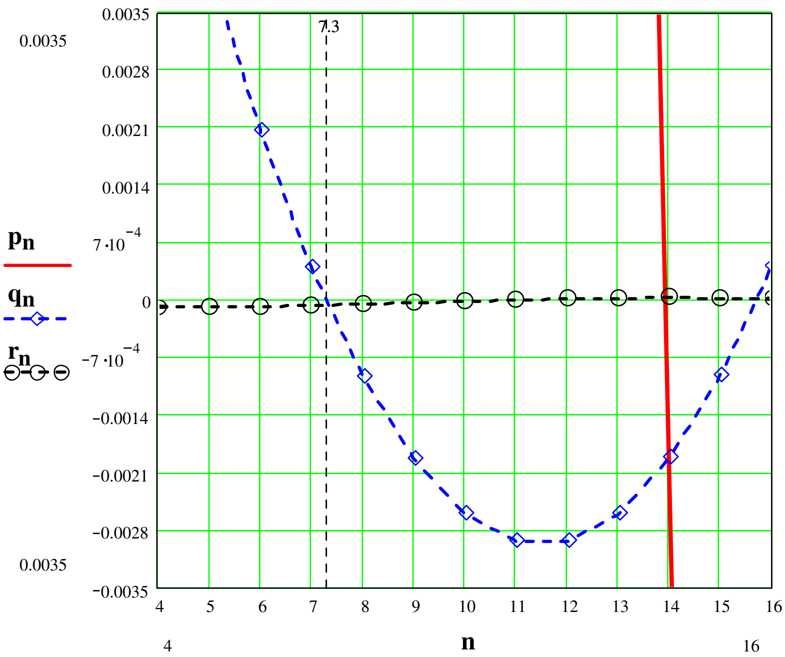

Fig. 1. Variation of the stability parameters with the number of fermion flavors.

[20]. The expectation is that, by using one or more of these scenarios, the actual number of SM flavors $n = 6$ may be recovered at the end of calculations.

It is also instructive to note that the condition $p(n) \geqslant 0$ determines the largest number of flavors that preserves the flow stability. From the graph we see that this number is $n_{\mathrm{MAX}} \approx 14$, consistent with the maximum number of quark flavors that maintains asymptotic freedom in QCD [4].

5. Conclusions

The origin of the six known generations of active fermions continues to be an unresolved issue of SM. We have examined in this work the possibility that the number of fermion generations is rooted in the stability of the gauge coupling flow. Constraining the coupling trajectories to settle on a set of isolated stationary points brings the number of flavors to seven. This result *either* makes room for an additional fermion generation in future tests of SM *or* suggests that our stability analysis is valid only up to a first-order approximation. The largest number of flavors for which the coupling trajectory remains stable was found to be fourteen. Future works on the topic may be devoted to the analysis of the gauge coupling flow in the presence of higher-order diagrams and/or random perturbations. A number of excellent studies exist on the subject of stochastic stability for multidimensional nonlinear systems. Although a complete listing is impractical, we believe that the methodology discussed in [22–25] may provide a suitable starting baseline.

Appendix A. On the Standard Model for particle physics [1–8,10,19]

SM combines relativity and quantum mechanics in a unified conceptual framework known as relativistic quantum field theory (QFT). Electromagnetic, weak and strong interactions are all included in SM and are described by abelian and non-abelian gauge theories. The structure of SM is a generalization of that of QED – the quantum theory of electromagnetic phenomena – to a larger set of conserved currents and charges. In SM the matter fields have spin 1/2 and are divided into two groups: quarks (the constituents of protons, neutrons and all hadrons) and leptons. There are six known generations (flavors) of quarks and six generations of leptons. There are eight color charges, which couple quarks in QCD and four electroweak charges, which couple leptons and quarks. All interactions are carried through gauge particles of spin 1. They are, respectively, the photon γ, the three vector bosons of the weak interaction W^+, W^-, Z^0 and the eight gluons of the strong interaction. The set of three interactions can be formulated in terms of unitary groups of different dimensions. It is customary to denote the gauge structure of SM as a product expressed as $SU(3) \times SU(2) \times U(1)$. This notation has the following meaning: a gauge theory described by the group $SU(N)$ is defined in terms of $N^2 - 1$ underlying gauge bosons. The group $SU(3)$ is the gauge group of QCD, which carries the $3^2 - 1 = 8$ gluons of the strong interaction. The $SU(2) \times U(1)$ group represents the structure of the electro-weak model with $2^2 - 1 = 3$ corresponding massive vector bosons, namely (γ, W^+, W^-, Z^0).

The interaction amplitude is determined by the magnitude of a coupling constant, generically denoted by g *or* by the magnitude of the coupling strength $\alpha = g^2/4\pi$. A QFT characterized by a dimensionless coupling constant $g \ll 1$ is said to be weakly coupled and it is well defined by an expansion in powers of g, called perturbation theory. Otherwise, the theory is said to be strongly coupled. Perturbation techniques have limited applicability in strongly coupled theories and various non-perturbative methods have to be implemented in order to derive meaningful results.

Appendix B. The renormalization theory of coupling flow

The underlying idea of renormalization is to avoid divergences that show up in physical predictions of QFT by using systematic rules for performing calculations [1,2,7,8,19]. In general, a QFT is called renormalizable if all infinities can be absorbed into a redefinition of a finite number of parameters. There are several technical procedures to renormalize a field theory. One standard way is to cut off the integrals in the calculations at a large but finite value of momentum (Λ). The renormalization is successful if, after taking the limit $\Lambda \to \infty$, the resulting quantities are finite and independent of Λ.

An important consequence of the renormalization program is that all parameters of the theory depend on the energy scale at which the phenomena are recorded (μ). The so-called *beta function* encodes the evolution of a given parameter with the energy scale. For instance, the coupling flow equation is defined by the relation

$$\mu \frac{\partial g}{\partial \mu} = \beta(g). \tag{B1}$$

If the beta-functions of a QFT vanish, then the theory approaches a so-called *fixed point* where it becomes *scale-invariant*. The coupling parameters of a quantum field theory can flow even if the corresponding classical field theory is scale-invariant. In this case, the non-vanishing beta function indicates that the classical scale-invariance is *anomalous*. If a beta-function is *positive*, the corresponding coupling increases with increasing energy. An example is QED, where one finds by using perturbation theory that the beta-function is positive. In particular, at low energies, the fine-structure constant measures $\alpha_{\mathrm{EM}} \approx 1/137$, whereas at the scale of the Z boson, about 90 GeV, the same constant becomes $\alpha_{\mathrm{EM}} \approx 1/127.9$. In non-abelian gauge theories, the beta function can be *negative*. An example is the beta-function for QCD whose coupling decreases at high-energies. Furthermore, the coupling decreases logarithmically, a phenomenon known as *asymptotic freedom*. This means that the coupling becomes large at low energies and predictions can no longer rely on perturbation theory.

Appendix C. The Routh–Hurwitz criterion

We review here implementation of the Routh–Hurwitz criterion in the case of a three-dimensional system of nonlinear differential equations. For additional details, the reader is referred to [18]. Consider the three-dimensional system:

$$\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + P_1(x_1, x_2, x_3),\\
\dot{x}_2 &= a_{12}x_1 + a_{22}x_2 + a_{23}x_3 + P_2(x_1, x_2, x_3),\\
\dot{x}_3 &= a_{13}x_1 + a_{23}x_2 + a_{33}x_3 + P_3(x_1, x_2, x_3),
\end{aligned} \tag{C1}$$

in which the functions P_i contain no linear terms. The characteristic equation of (C1) takes the form of the cubic polynomial

$$\lambda^3 + p\lambda^2 + q\lambda + r = 0, \tag{C2}$$

where the three stability parameters are given by

$$\begin{aligned} p &= -(a_{11} + a_{22} + a_{33}), \\ q &= \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{31} \\ a_{13} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix}, \\ r &= -\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}. \end{aligned} \tag{C3}$$

The Routh–Hurwitz stability condition amounts to the following condition:

$$p > 0,\ q > 0,\ r > 0 \quad \text{and } R \equiv pq - r > 0. \tag{C4}$$

Boundaries of the stability region are defined by two surfaces ($r = 0, p > 0, q > 0$) and ($R = 0, p > 0, q > 0$). Eq. (C2) has at least one vanishing root on the surface $r = 0$, and a pair of imaginary roots on the surface ($R = 0, q > 0$).

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